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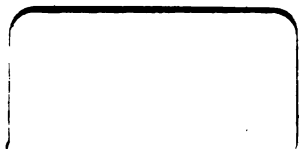


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MATHEMATICAL MONTHLY.

Vol. III

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Oct. 1860 - May 1861

OCTOBER, 1860.

EDITED BY

J. D. RUNKLE, A.M., A.A.S.

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Editorial Items.

Prizes for Vol. III. offered to all Students.

FOR SOLUTIONS. — A 1st Prize, \$7.50; 2d, \$5; 3d, \$3; 4th, Vol. III. The third and fourth prizes are confined to beginners for solutions of Problems I. and II.; but as the best solutions only will be published, advanced students are desired to send solutions of all the problems.

FOR ESSAYS. — A 1st Prize, \$50; 2d, \$40; 3d, \$30; 4th, \$20; 5th, \$10. The first and second of these prizes are open to all competitors. Instead of cash prizes, we propose to make them all payable in mathematical books. This change will make the prizes less burdensome to us while it will be appropriate, and we believe quite as acceptable to competitors, to receive standard mathematical books, — such as BOWDITCH's edition of *Mécanique Céleste*, PEIRCE's *Analytic Mechanics*, BARTLETT's *Analytic Mechanics*, DAVIS's edition of *Theoria Motus*, DAVIES and PECK's *Mathematical Dictionary*, CHAUVENET's *Analytic Trigonometry*. But selections may be made from SEVER & FRANCIS's list, and desired works not found there will be ordered. To those preferring standard works in other departments, we offer WORCESTER's *Quarto Dictionary* and the fine editions of HUME, GIBBON, LAMB, and MACAULAY, described

in our advertising pages. To this list we may hereafter add editions of other standard authors, if desired.

One Plane Descriptive Geometry.

Mr. WATSON writes us that his work on this subject (about fifty pages of text and twelve plates) is nearly ready. The plates will be engraved in Paris by the same engraver, and printed on the same quality of paper, as those in LEROY's *Descriptive Geometry*. A supplement will contain an application to a road constructed in France by the Corps des Ponts et Chaussées.

Autograph Note of Laplace.

Mr. WATSON has sent us lithograph copies of a note in the handwriting of the distinguished Geometer, which will appear in our next.

Portrait of Sir John Herschel.

Through the kindness of Miss MARIA MITCHELL we have received a fine photograph of Sir JOHN HERSCHEL, which we shall have engraved on steel for an early number.

We expect our friends will aid us by increasing our subscription list. See *Mathematical Monthly*, on seventh page of Advertiser.

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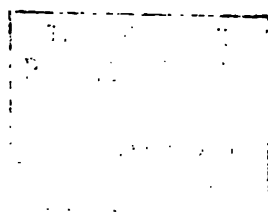
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J. F. W. Herschel

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THE

MATHEMATICAL MONTHLY.

Vol. III . . . OCTOBER, 1860. . . No. I

PRIZE PROBLEMS FOR STUDENTS.

I. Let AB and CD be two diameters of a given circle, drawn at right angles to each other; AEB a circular arc described with radius DA or DB . Prove that the area of the lune $AEB C$ equals the area of the triangle ADB .

II. Of all isoperimetrical polygons having the same number of sides, the greatest is that which is equilateral.

III. In any triangle, if P , P' , and P'' denote the perpendiculars from the vertices to the opposite sides; p , p' , and p'' the portions of the same between their point of intersection and the sides. Prove that

$$\frac{p}{P} + \frac{p'}{P'} + \frac{p''}{P''} = 1.$$

— Communicated by J. A. WINEBRENER, Princeton College, N. J.

IV. At what angle must the rudder of a vessel be inclined to the stream, that the effect produced may be a maximum?

V. Two circles, radii r and r_1 , have their centres united by a line d feet long. Required the position upon this line of the centre of a third circle, which shall cut off a square feet from the first circle and b feet from the second. — Communicated by ASHER B. EVANS.

Solutions of these problems must be received by December 1, 1860. (See Editorial Items.)

NOTE* ON SOME PROPERTIES OF SQUARE NUMBERS.

By PROF. D. W. HOYT, Fairfax, Vt.

IF all *differences* of square numbers which produce squares are known, the *sums* of squares which produce squares are also known; for the one is obtained from the other by the transposition of a single term of the equation.

The differences between consecutive squares of the series 0, 1, 4, 9, 16, 25, &c., are 1, 3, 5, 7, 9, &c., a series including all the odd numbers, and consequently the squares of all odd numbers. The square of every odd number is, therefore, equal to the difference between the squares of two numbers differing by unity. 1 is the difference between 0 and 1; the second odd square is 9, which is the difference between 16 and 25; the third is 25, which is the difference between 144 and 169; that is, $1^2 = 1^2 - 0^2$, $3^2 = 5^2 - 4^2$, and $5^2 = 13^2 - 12^2$, from which other equations of the series can readily be obtained, as the common difference of the first members is 2, and the terms of the second members have 4 as a common difference of the second order.

The differences between alternate terms of the series of square numbers are 4, 8, 12, 16, &c., including all numbers divisible by 4; hence the square of every even number is equal to the difference between the squares of two numbers differing by 2. By an inspection of these series, we obtain $2^2 = 2^2 - 0^2$, $4^2 = 5^2 - 3^2$, $6^2 = 10^2 - 8^2$, &c.

The differences between the squares of numbers differing by 3 include all odd numbers divisible by 3, and consequently all those divisible by 9; hence the square of every odd number divisible by 3

* This note was furnished by PROF. HOYT, in connection with that published on pp. 264, 265, May, 1860, but was then omitted for want of space. — ED.

is equal to the difference between the squares of two numbers differing by 3. The differences between the squares of numbers differing by 4 include all numbers divisible by 8, and consequently all those divisible by 16, or the squares of those divisible by 4. The use of numbers differing by 5, 6, and 7, as well as 3 and 4, give equations which are multiples of those contained in the first two series; but 8 and 9 introduce new relations of numbers. The differences between the squares of numbers differing by 8, in the series $(-3)^2$, $(-2)^2$, $(-1)^2$, 0^2 , 1^2 , 2^2 , 3^2 , 4^2 , 5^2 , 6^2 , &c., are 16, 32, 48, 64, 80, &c., including all numbers divisible by 16; and consequently the squares of all numbers divisible by 4, giving a series of equations whose first members are the same as when the numbers differ by 4. The square of every odd number divisible by 3 is also equal to the difference between the squares of two numbers differing by 9, as well as two others differing by 3.

In general, if m represent any odd number whatever, then the square of every odd number divisible by m will be equal to the difference between the squares of two numbers differing by m . If n represent any even number, the square of every number divisible by n will be equal to the difference between the squares of two numbers differing by n . If m contains a square factor, or if n contains a square factor in addition to the factor 2, there will also be numbers not divisible by m or n whose squares will be equal to the differences between the squares of numbers differing by m or n . The following statement, however, includes all possible cases: Every odd square divisible by m will be equal to the difference between the squares of two numbers differing by m , and every square divisible by $2n$ will be equal to the difference between the squares of two numbers differing by n .

Since $18 = 2 \times 3^2$ and $25 = 5^2$, the use of those numbers as differences will introduce some equations not multiples of any preceding ones.

The same results may be more readily and satisfactorily reached by another process. The general equation is of the form $M^2 = N^2 - P^2$. Let $N - P = p$, or $P = N - p$, and the general equation becomes $M^2 = 2pN - p^2$. As this equation contains three variable quantities, we may give any values whatever to two of them, and the third will be a function of those two. But if we consider either M or p as the function, the equation will assume the quadratic form, and we cannot attribute all values to the arguments without producing surds. It will therefore be best to consider M and p the arguments, and N the function, or $N = \frac{M^2 + p^2}{2p}$. If we attribute any rational values whatever to M and p , the values of N and $N - p$ will be rational; but our present purpose is to deal only with whole numbers, and we can, therefore, only substitute such integers for M and p as shall render N a whole number. It is evident that any whole number may be used for p , and that M^2 must in all cases be divisible by p . Since $N = \frac{1}{2}(\frac{M^2}{p} + p)$, $\frac{M^2}{p} + p$ must be an even number; consequently if p is an odd number, $\frac{M^2}{p}$ must be odd, that is, M^2 may be any odd square divisible by p ; and if p is an even number, $\frac{M^2}{p}$ must be even, that is, M^2 may be any square divisible by $2p$.

If we make $p = 1$, and regard it as constant, M may be any odd number whatever. When $p = 2$, M^2 must be divisible by 4, and M may be any even number whatever. If $p = 3$, M may be any odd number divisible by 3, and so on. For each value of p we have an infinite series of equations; and since p may have an infinite number of values, we have an infinite number of infinite series. As p is constant for any given series, N and $N - p$ may be readily found by the above formula for any possible value of M . When three equations are thus obtained, the law of the series becomes evident, and the other equations may be obtained without the aid of the formula.

It is worthy of notice that each series, in descending, returns upon itself, — except that M becomes negative, — and each series is thus unlimited in both directions. If the series where $p = 8$ be continued backwards, we have $0^2 = 4^2 - (-4)^2$, $(-4)^2 = 5^2 - (-3)^2$, $(-8)^2 = 8^2 - 0^2$, $(-12)^2 = 13^2 - 5^2$, &c. The formula also shows that M may be made either positive or negative without any change in the other quantities.

As $M^2 + p^2$ must always be positive, it is evident that if we make p negative, N must also be negative (M being either positive or negative, as before), and the series will differ from those produced by the corresponding positive values of p only in having N and $N - p$ take an opposite sign. When $p = 0$, $N = \infty$, and $M^2 = \infty^2 - \infty^2$, unless M is also 0, when $0^2 = N^2 - N^2$. In the two equations last named, M and N represent any finite quantity whatever.

The following table contains the most interesting and important of the first thirty series. Others, up to the thirty-second (where $p = 32$), are merely multiples of those given.

$p = m = 1.$	$p = n = 8.$	$p = n = 18.$
$1^2 = 1^2 - 0^2$	$4^2 = 5^2 - (-3)^2$	$6^2 = 10^2 - (-8)^2$
$3^2 = 5^2 - 4^2$	$8^2 = 8^2 - 0^2$	$12^2 = 13^2 - (-5)^2$
$5^2 = 13^2 - 12^2$	$12^2 = 13^2 - 5^2$	$18^2 = 18^2 - 0^2$
$7^2 = 25^2 - 24^2$	$16^2 = 20^2 - 12^2$	$24^2 = 25^2 - 7^2$
$9^2 = 41^2 - 40^2$	$20^2 = 29^2 - 21^2$	$30^2 = 34^2 - 16^2$
$11^2 = 61^2 - 60^2$	$24^2 = 40^2 - 32^2$	$36^2 = 45^2 - 27^2$
$13^2 = 85^2 - 84^2$	$28^2 = 53^2 - 45^2$	$42^2 = 58^2 - 40^2$
	$32^2 = 68^2 - 60^2$	$48^2 = 73^2 - 55^2$
$p = n = 2.$	$36^2 = 85^2 - 77^2$	$54^2 = 90^2 - 72^2$
$2^2 = 2^2 - 0^2$	$p = m = 9.$	$p = m = 25.$
$4^2 = 5^2 - 3^2$	$3^2 = 5^2 - (-4)^2$	$5^2 = 13^2 - (-12)^2$
$6^2 = 10^2 - 8^2$	$9^2 = 9^2 - 0^2$	$15^2 = 17^2 - (-8)^2$
$8^2 = 17^2 - 15^2$	$15^2 = 17^2 - 8^2$	$25^2 = 25^2 - 0^2$
$10^2 = 26^2 - 24^2$	$21^2 = 29^2 - 20^2$	$35^2 = 37^2 - 12^2$
$12^2 = 37^2 - 35^2$	$27^2 = 45^2 - 36^2$	$45^2 = 53^2 - 28^2$
$14^2 = 50^2 - 48^2$	$33^2 = 65^2 - 56^2$	$55^2 = 73^2 - 48^2$
$16^2 = 65^2 - 63^2$	$39^2 = 89^2 - 80^2$	$65^2 = 97^2 - 72^2$
$18^2 = 82^2 - 80^2$		

NOTE ON THE "HORIZONTAL THRUST OF EMBANKMENTS."

By PROF. F. W. BARDWELL, Antioch College, Yellow Springs, Ohio.

I HAVE taken the earliest opportunity since my return to examine more carefully the question of the horizontal thrust of an embankment of earth rising from the retaining wall in a natural slope, and I am happy to find that my former criticism upon an article by Captain WOODBURY was entirely at fault, and that his formula was correct.

I think I understood in the first place the conditions of the problem, but was misled by relying, without sufficient examination, upon a statement in the American edition of WEISBACH's *Mechanics*. The question of the horizontal thrust of an embankment terminating in a horizontal plane is there discussed at length, and the usual formula deduced. But again (Vol. II., page 18, *Remark*) it is stated that this same formula is applicable to the case where the earth is carried above the wall in a natural slope, provided the height of the earth be substituted in the formula for that of the wall. At first this seemed plausible; and, accepting it without further scrutiny, I was afterwards led to criticise too hastily the article of Captain WOODBURY.

The formula given by WEISBACH is,

$$P = \frac{1}{2} h^2 \gamma \tan^2 \left(45^\circ - \frac{\phi}{2} \right);$$

P being the horizontal thrust, h the height of the wall, γ the density of the earth, and ϕ the angle of friction. To show that this formula is not applicable in the case of a sloping surface, by making h equal to the height of the earth, let us suppose the simple case where the height of the wall is zero, and the earth extends back indefinitely with its natural slope. Now the horizontal thrust is obviously zero, but the above formula would make it indefinitely great.

I have quoted from the American edition of WEISBACH, which contains some additions to the original ; but I have not a copy of the German edition at hand to consult with reference to this point.

Let me add, with regard to the paper of Captain WOODBURY, that a careful examination shows the elegance of the method he used, and the simplicity of the formula deduced for its practical application.

NOTE ON THE DEPENDENCE OF EQUATIONS.

By JOHN BORDON, Chicago, Illinois.

A METHOD of deducing equations of a higher order from those of a less degree, by a method of combining the imaginary roots of $x^3 - a^3$, $x^5 - a^5$, &c., has been heretofore shown. The results thereby arrived at can be more easily obtained in some cases by the law of symmetrical functions. For, suppose we have an equation, $x^2 + bx + c = 0$, and we find the coefficients of another equation of the same degree whose roots shall be the cubes of those in the above equation ; and suppose $x^2 + Bx + C = 0$ to be the equation ; then if $-a$, $-b$, are the roots of the first equation, then $B = a^3 + b^3$, $C = a^3 b^3$. But by symmetrical functions, if we represent by S_1 , S_2 , S_3 , &c., the sum of the first power, sum of the squares, sum of the cubes, &c., of the roots of the first equation, we have

$$S_1 = -b ; S_2 = b^2 - 2c ; S_3 = -b^3 + 3bc.$$

But $B = -S_3 = +b^3 - 3bc$, and since $c = a b$, evidently $C = a^3 b^3$, or $c = C^{\frac{1}{3}}$. Hence we have

$$b^3 - 3 C^{\frac{1}{3}} b - B = 0.$$

Here we have an equation of the third degree deduced out of two others of the second degree, and by this means we can arrive at

the CARDAN formula. For if the above cubic be the same as $x''^3 + p x'' + q = 0$, then $x'' = b$, $p = -3 C^2$, $q = -B$, or $C = -\frac{p^2}{27}$, $B = -q$, and we have

$$x'^2 - qx - \frac{p^3}{27} = 0, \quad \text{or} \quad x' = +\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}};$$

but $x' = x^2$, or $x = x'^{\frac{1}{2}}$, and $b =$ sum of the two values of x taken with contrary signs; and as

$$-x = \left(-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{2}},$$

then

$$x'' = b = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{2}} + \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{2}}.$$

So, if we went on obtaining the values of S_4, S_5, S_6, S_7 , we would obtain equations of the 4th, 5th, 6th, and 7th degrees, which would be resolvable in the same manner. These would give the forms of equations of those degrees which depend for their solution upon the quadratic. The two modes of solving biquadratics can be arrived at also by symmetrical functions, but the manner of doing so, heretofore pointed out, cannot be simplified. When we arrive at an equation of the 5th degree of the general form $x^5 + p x^3 + \&c.$, the mode of solving cubics and biquadratics by the use of two equations of the next lower degree becomes more difficult of application. But the equation $x^5 + p x^3 + \&c.$ can be deduced from two equations of the fourth degree. And it yet remains to be seen whether a solution of the quintic cannot be fairly looked for in that direction. And the general proposition may be stated, that an equation of the form $x^m + p x^{m-2} + \&c. = 0$ may be derived from two equations of the form

$$(1) \quad x'^{m-1} + A x'^{m-2} + \&c = 0,$$

$$(2) \quad x''^{m-1} + a x''^{m-2} + \&c = 0;$$

which shall stand to each other in the relation that the roots of Eq. (1) are the m th power of the roots of Eq. (2), i. e. $x_{,,}^m = x_{,}$.

And to illustrate this, let us deduce $x^5 + p x^3 + q x + r = 0$ from two biquadratics. First, we have

$$(3) \quad x_{,}^4 + A x_{,}^3 + B x_{,}^2 + C x_{,} + D = 0,$$

$$(4) \quad x_{,,}^4 + a x_{,,}^3 + b x_{,,}^2 + c x_{,,} + d = 0;$$

in which we suppose $x_{,} = x_{,,}^5$. Now let us arrive at the value of A in terms of a by symmetrical functions :—

$$S_1 = -a; \quad S_2 = a^2 - 2b; \quad S_3 = -a^3 + 3ab - 3c;$$

$$S_4 = a^4 - 4a^2b + 4ac + 2b^2 - 4d;$$

$$S_5 = -a^5 + 5a^3b - 5a^2c - (5b^2 - 5d)a + 5bc.$$

But $A = -S_5 = a^5 - 5a^3b + 5a^2c + (5b^2 - 5d)a - 5bc.$

Hence

$$(6) \quad a^5 - 5ba^3 + 5ca^2 + (5b^2 - 5d)a - 5bc - A = 0.$$

Hence, if we have

$$(7) \quad x^5 + p x^3 + q x^2 + r x + s = 0,$$

then $a = x$; $p = -5b$; $q = 5c$; $r = 5b^2 - 5d$; $S = -A - 5bc$; that is to say, a root of an equation of the 5th degree, as Eq. (7), will be equal to the sum of the roots of an equation of the fourth degree, as Eq. (4), taken with a contrary sign, when $b = -\frac{p}{5}$; $c = \frac{q}{5}$; $d = \frac{p^2 - 5r}{25}$; $A = \frac{pq}{5} - S$. Now D in Eq. (3) is equal to d^5 , hence $D = \left(\frac{p^2 - 5r}{25}\right)^5$; and if B and C were determined in like manner, then x in Eq. (7) would be equal to the sum of the 5th roots of Eq. (3) taken with their signs changed, and the quintic would be resolved into the biquadratic. The law of symmetrical functions renders the general proposition obvious.

NOTE ON NUMERICAL EQUATIONS.

By PROF. A. VALLAS, State Seminary and Military Academy, Alexandria, Louisiana.

THE Resolution of Numerical Equations, so far at least as real roots are concerned, has been brought to such perfection that scarcely more is expected at present than to render approximations more easy and expeditious.

It is known that the common method of finding the irrational real roots of equations is still that of NEWTON, although considerably amended and perfected by such men as BUDAN, FOURIER, and HORNER.

Geometrically this common method consists in this. The equation being referred to two rectangular axes, and the real roots being the abscissas of the intersections of the horizontal axis by the curve, from an ordinate and trigonometrical tangent of a point near an intersection, to find this intersection by means of the "subtangent."

For in the equation

$$fx = x^m + Ax^{m-1} + Bx^{m-2} + \dots + Kx + L = 0,$$

put $a + x'$ for x , the results will be,

$$f(a + x') = fa + f'a \cdot x' + \frac{f''a}{1 \cdot 2} x'^2 + \dots + \frac{f^{(m)}a}{1 \cdot 2 \dots m} x'^m = 0.$$

If now a be near one of the real roots, we get, in general, a still nearer value by putting

$$fa + f'a \cdot x' = 0,$$

from which

$$x' = -\frac{fa}{f'a};$$

and the nearer value will be

$$a + x' = a - \frac{fa}{f'a}.$$

That here x' is the subtangent of the point whose abscissa is a , needs not to be proved.

How that, in order to a sure approximation, the root ought first to be included between such limits as contain no singular points, as also how the degree of approximation to be obtained may be determined beforehand, has, with other particulars, been admirably discussed by *FOURIER*; although it must be regretted that *FOURIER* despised to connect his disquisitions with those of *BUDAN*, whose transformed equations (or equations where roots are all less by a) afford very easy means for determining the numerical values of

$$f a, f' a, f'' a, f''' a \dots,$$

which form the substratum of *FOURIER*'s discussion. The two proceedings were at last connected by others, and a method engrafted on this connection, namely, also by *HORNER*, which may be regarded as the triumph of modern investigation in these matters.

It were needless here to do more than merely mention, that the diminution of roots is, geometrically, equivalent to transposing the axis of the ordinates to the right hand, and that according to this the approximations are obtained by subtangents belonging to points of the curve situated in that axis.

In this note it is proposed to show : 1st, that the approximations aimed at may be more easily obtained by abscissas of parabolic arcs tangent to the curve where it intersects the axis of the ordinates ; and, 2d, to give a sure rule (as *FOURIER* has done for *NEWTON*'s method) for the degree of approximation thus obtained.

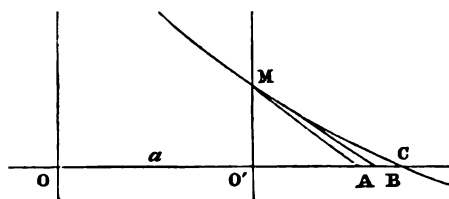
In regard to the first point, let

$$f(a + x') = f a + f' a \cdot x' + \frac{f'' a}{1 \cdot 2} \cdot x'^2 + \dots = 0$$

be such a transformed equation as excludes all singular points between the point belonging to a and the nearest intersection, and put

$$f a + f' a \cdot x' + \frac{f'' a}{1 \cdot 2} \cdot x'^2 = 0;$$

then will this latter equation manifestly belong to a parabolic curve, having a contact of the second order, at the point belonging to a , with the curve represented by the equation $fx = 0$, or having that contact in the axis of the ordinates, as this axis is supposed to have been removed by as much as a to the right hand.



In the annexed figure, O being the origin of the co-ordinates of the curve represented by $y = fx$, O' will be that of the curve $y' = f(a + x')$. Further, M being the point of the curve

in the new axis of the co-ordinates, $O'A$ is the approximation of the nearest intersection by the subtangent, while $O'B$ is that by the abscissa of the parabolic arc in question.

Solving the equation

$$fa + f'a \cdot x' + \frac{f''a}{2} x'^2 = 0,$$

we obtain
$$x' = -\frac{f'a}{f''a} \pm \frac{f'a}{f''a} \sqrt{1 - \frac{2fa \cdot f''a}{(f'a)^2}},$$

in which expression the upper sign is to be taken; for, 1st, if x' admits values of different signs, the conditions of the problem exclude the negative one; and, 2d, if both should be positive, those same conditions require us to take the less.

Developing now the expression for x' , just found, we get

$$x' = -\frac{f'a}{f''a} - \left(\frac{f'a}{f''a}\right)^2 \cdot \frac{f''a}{2f'a} - 2\left(\frac{f'a}{f''a}\right)^3 \cdot \left(\frac{f''a}{2f'a}\right)^2 \dots$$

This is the expression for the abscissa of the parabolic arc, or for $O'B$, which abscissa is evidently made up of the subtangent $O'A$ represented by the first, and the portion AB represented by the succeeding terms of the development.

Now, in regard to the degree of approximation, it is evident that in developing in a similar way the value of x'' from

$$f a + f' a \cdot x'' + \frac{f'' (a + \theta x'')}{2} \cdot x''^2 = 0,$$

which is manifestly the given equation itself; we obtain for $x'' = O' C$,

$$O' C = -\frac{f a}{f' a} - \left(\frac{f a}{f' a}\right)^2 \cdot \frac{f'' (a + \theta x'')}{2 f' a} - 2 \left(\frac{f a}{f' a}\right)^3 \cdot \left(\frac{f'' (a + \theta x'')}{2 f' a}\right)^2 \dots$$

Or, developing $f'' (a + \theta x'')$,

$$\begin{aligned} O' C = & -\frac{f a}{f' a} - \left(\frac{f a}{f' a}\right)^2 \cdot \frac{f'' a}{2 f' a} - 2 \left(\frac{f a}{f' a}\right)^3 \cdot \left(\frac{f'' a}{2 f' a}\right)^2 \dots \\ & - \theta x'' \cdot \left(\frac{f a}{f' a}\right)^2 \cdot \frac{f''' a}{2 f' a} \dots \end{aligned}$$

Leaving out the other terms, we see easily that the difference of the two abscissas $O' C - O' B = B C$ is

$$\begin{aligned} &= -\theta x'' \left(\frac{f a}{f' a}\right)^2 \cdot \frac{f''' a}{2 f' a}, \\ &= -3 \theta x'' \cdot \left(\frac{f a}{f' a}\right)^2 \cdot \frac{f''' a}{2 \cdot 3 f' a}; \end{aligned}$$

and this expression we shall directly discuss.

Suppose the transformed equation to be

$$x'^m + B_{m-1} x'^{m-1} + B_{m-2} x'^{m-2} + \dots + B_3 x'^3 + B_2 x'^2 + B_1 x' + B_0 = 0;$$

then we shall have

$$B_0 = f a, \quad B_1 = f' a, \quad B_2 = \frac{f'' a}{2}, \quad B_3 = \frac{f''' a}{2 \cdot 3} \dots$$

Substituting these values, the difference between the abscissa of the curve represented by the given equation and that of the parabolic arc, or

$$B C \text{ will be } = -3 \theta x'' \left(\frac{B_0}{B_1}\right)^2 \cdot \frac{B_3}{B_1}.$$

If now
$$\frac{B_2}{B_1} = \frac{1}{10^m},$$

and we suppose a to be a portion of the root with n correct decimals, that is, x'' , and (which is very nearly the same)

$$\frac{B_0}{B_1} = \frac{1}{10^n};$$

we see easily that the difference $OC - OB = BC$ will be numerically

$$= \frac{3}{10^{m+3n}},$$

and we get therefore, in all cases, through the abscissa of the parabolic arc, at least $m + 3n - 1$ correct decimal places.

Let us now return for a moment to the development of this abscissa.

$$OC = -\frac{fa}{f'a} - \left(\frac{fa}{f'a}\right)^2 \cdot \frac{f''a}{2f'a} - 2 \left(\frac{fa}{f'a}\right)^3 \cdot \left(\frac{f''a}{2f'a}\right)^2 \dots$$

By a similar substitution, the third term becomes

$$- 2 \left(\frac{B_0}{B_1}\right)^3 \cdot \left(\frac{B_2}{B_1}\right)^2.$$

If we now put
$$\frac{B_2}{B_1} = \frac{1}{10^\mu},$$

we shall have
$$2 \left(\frac{B_0}{B_1}\right)^3 \cdot \left(\frac{B_2}{B_1}\right)^2 < \frac{2}{10^{2\mu+3n}};$$

hence this term can only affect the $(2\mu + 3n)$ th decimal, and may therefore be neglected, when even

$$2\mu \leq m - 1.$$

Having therefore, by diminutions of the roots, sufficiently approached one of the irrational real roots, and having thus obtained a transformed equation of the form

$$x'^m + B_{m-1} x'^{m-1} + \dots + B_3 x'^3 + B_2 x'^2 + B_1 x' + B_0 = 0,$$

we shall find $x' = -\frac{B_0}{B_1} - \left(\frac{B_0}{B_1}\right)^2 \cdot \frac{B_2}{B_1} - 2 \left(\frac{B_0}{B_1}\right)^3 \cdot \left(\frac{B_2}{B_1}\right)^2$

for an approximation correct in $m + 3n - 1$ decimals, if by previous operations n have been found already. And in this it must be remarked, 1st, that m is to be determined by the relation,

$$\frac{B_3}{B_1} < \frac{1}{10^m};$$

and that, 2d, the third term of the value of x' may be omitted whenever $2\mu \leq m$, μ being determined by the relation,

$$\frac{B_2}{B_1} < \frac{1}{10^\mu}.$$

Let us take, for an example, the equation

$$x^3 - 10x^2 - 130x + 858 = 0.$$

It is easy to find that the real roots are contained between 5 and 6, 14 and 15, —10 and —11. In order to find the first of these roots, let us diminish all roots by 5.

$$\begin{array}{r} 5) 1 \quad -10 \quad -130 \quad +850 \\ \quad 1 \quad -5 \quad -155 \quad +75 \\ \quad 1 \quad 0 \quad -155 \\ \hline x^3 + 5x^2 - 155x' + 75 = 0. \end{array}$$

It is easy to find the first decimal. According to the common method, there is sometimes a doubt about this. According to our proceeding,

$$-\frac{B_0}{B_1} = \frac{75}{155} = 0.48; \quad -\left(\frac{B_2}{B_1}\right)^2 \cdot \frac{B_2}{B_1} = \left(\frac{75}{155}\right)^2 \cdot \frac{5}{155} = 0.00;$$

and there is therefore no doubt about this first decimal. Diminishing by 0.4,

$$\begin{array}{r} 0.4) 1 \quad 5 \quad -155 \quad 75 \\ \quad 1 \quad 5.4 \quad -152.84 \quad 13.864 \\ \quad 1 \quad 5.8 \quad -150.52 \\ \hline x'^3 + 6.2x'^2 - 150.52x'' + 13.864 = 0. \end{array}$$

Before proceeding further, let us examine how many decimals we shall get by the application of our proceeding.

$$\frac{B_3}{B_1} = \frac{1}{150} < \frac{1}{10^2}, \quad \frac{B_2}{B_1} = \frac{6}{150} < \frac{1}{10}.$$

We have, therefore, $m=2$, $\mu=1$. This will lead us to the fourth decimal inclusively. We shall have

$$\begin{aligned} -\frac{B_0}{B_1} &= 0.09210 \\ -\left(\frac{B_0}{B_1}\right)^2 \cdot \frac{B_2}{B_1} &= \frac{0.00034}{0.0924} \end{aligned}$$

The next approximation would give us thirteen correct decimals. If we diminish, however, only by 0.09, we shall obtain at least seven, and this we shall do.

$$\begin{array}{r r r r} 0.09) 1 & 6.2 & - 150.52 & 13.864 \\ & 9 & \underline{0.5661} & - 13.495851 \\ & 6.29 & - 149.9539 & \underline{0.368149} \\ & 9 & \underline{0.5742} & \\ & 6.38 & - 149.3797 & \\ & 9 & & \end{array}$$

$$x''^3 + 6.47 x''^2 - 149.3797 x''' + 0.368149 = 0;$$

from which $-\frac{B_0}{B_1} = 0.00246 \ 451$

$$-\left(\frac{B_0}{B_1}\right)^2 \cdot \frac{B_2}{B_1} = \frac{0.00000 \ 026}{0.00246 \ 478}$$

The root with seven correct decimals is, therefore, 5.4924647.

In applying this method to binomial equations, $x^n + N = 0$, put $x' + a$, instead of x :

$$\begin{aligned} x'^n + n a x'^{n-1} + \frac{n(n-1)}{1 \cdot 2} a^2 x'^{n-2} + \dots \\ + \frac{n(n-1)}{1 \cdot 2} a^{n-1} x' + (a^n + N) = 0. \end{aligned}$$

As here $B_0 = a^n + N$, $B_1 = n a^{n-1}$, $B_2 = \frac{n(n-1)}{1 \cdot 2} a^{n-2} \dots$,
we get as an approximate value,

$$x' = -\frac{1}{n} \left(a + \frac{N}{a^{n-1}} \right) - \frac{(n-1)}{2a} \left[\frac{1}{n} \left(a + \frac{N}{a^{n-1}} \right) \right]^2 \\ - \frac{(n-1)^2}{2a^2} \left[\frac{1}{n} \left(a + \frac{N}{a^{n-1}} \right) \right]^3 \dots$$

And we get again at least $m + 3n' - 1$ correct decimals, after n' have been found, m being determined by the relation

$$\frac{(n-1)(n-2)}{1 \cdot 2} \cdot \frac{1}{a} < \frac{1}{10^m}.$$

If $n=2$, we obtain the following for the extraction of the square root from the number $-N$:

$$x' = -\frac{1}{2} \left(a + \frac{N}{a} \right) - \frac{1}{8a} \left(a + \frac{N}{a} \right)^2 - \frac{1}{16a} \left(a + \frac{N}{a} \right)^3 \dots$$

In this case, the approximation is unlimited, as $\frac{(n-1)(n-2)}{2 \cdot 3} = 0$; that is $m = \infty$.

If, for instance, $N = -12$, and $a = 3.46$, we get from the first three terms,

$$\begin{aligned} -\frac{1}{2} \left(a + \frac{N}{a} \right) &= + 0.00410 \ 40462 \ 4 \\ -\frac{1}{8a} \left(a + \frac{N}{a} \right)^2 &= - \quad \quad \quad 24339 \ 8 \\ -\frac{1}{16a} \left(a + \frac{N}{a} \right)^3 &= + \quad \quad \quad 28 \ 8 \\ \hline x' &= + 0.00410 \ 16151 \ (4); \end{aligned}$$

hence the square root of 12 is 3.46410 16151.

It may be remarked here, that the first term of the development of x' , namely, $-\frac{1}{n} \left(a + \frac{N}{a^{n-1}} \right)$, leads to an old rule, according to which, in the extraction of the n th root, nearly twice as many figures are to be found as are contained in a , and which rule is the follow-

ing: "Divide the number by the $(n - 1)$ st power of the part found (a) , subtract the quotient from a , and divide the difference by n ." It may also be remarked, that the same first term, as applied to the extraction of the square root, gives this root in nearly twice as many figures by the formula,

$$-\frac{a^2 + N}{2a} = -\frac{a^2 - N'}{2a} = \frac{N' - a^2}{2a},$$

(as N is supposed to be essentially negative,) which is nothing but a well-known rule, although commonly overlooked.

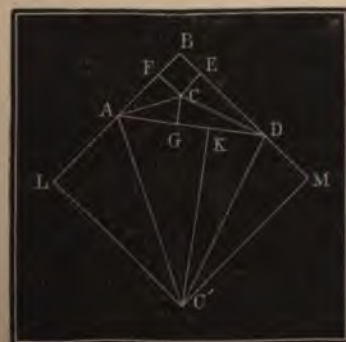
REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. IX., Vol. II.

The first Prize is awarded to ASHER B. EVANS, Madison Univ., N. Y.
The second prize is awarded to O. B. WHEELER, Michigan University.
The third prize is awarded to J. B. DAVISON, Oberlin College, Ohio.

PRIZE SOLUTION OF PROBLEM I.

By Miss HARRIET S. HAZELTINE, Worcester, Mass.

I. The sum of the radii of the inscribed circle and of the circle touching the hypotenuse and extension of the two legs of a right triangle is equal to the sum of the said legs.



Find C and C' , the centres of the two circles, by bisecting the angles including the hypotenuse AD , and their supplements. Then CE , CF , CG , perpendiculars to BD , AB , AD , will be radii of the inscribed circle, and $C'K$, $C'L$, $C'M$, perpendiculars to AD and to AB and BD prolonged, will be radii of the circle

whose centre is C' . $C'MBL$ is a square; $\therefore$ radius $C'M = AB + AL = AB + AK = AB + ED$ (Vol. II., No. VI., Problem II.). $CEBF$ is also a square; $\therefore$ radius $CE = EB$, and

$$C'M + CE = AB + ED + EB = AB + BD.$$

PRIZE SOLUTION OF PROBLEM II.

By MISS HARRIET S. HAZELTINE, Worcester, Mass.

II. The side of a regular octagon, inscribed in a circle whose radius is unity, is $\sqrt{2 - \sqrt{2}}$, and its area is $2\sqrt{2}$. Required the proof.

Denote by AB the side of the inscribed square, and draw the radii CA, CB to its extremities. Draw also the radius CDE perpendicular to and bisecting AB at D , and the subtended arc at E . The chord AE is the side of the regular inscribed octagon. $AB = \sqrt{2}$; $AD = \frac{1}{2}AB = \frac{1}{2}\sqrt{2}$. The angle $CAD = ACD \therefore CD = AD = \frac{1}{2}\sqrt{2}$. $DE = CE - CD = 1 - \frac{1}{2}\sqrt{2}$. $AE = \sqrt{AD^2 + DE^2} = \sqrt{(2 - \sqrt{2})}$. Four times the area of the triangle $ABE = 4 \times AD \times DE = 2\sqrt{2} - 2$. Adding this to the area of the inscribed square, which is 2, we shall obtain the area of the inscribed octagon, which is $2\sqrt{2}$.



PRIZE SOLUTION OF PROBLEM III.

By GUSTAVUS FRANKENSTEIN, Springfield, Ohio.

III. The perpendicular to the normal of a parabola at its intersection with the principal axis, meets the curve in two points through which tangents are drawn; find the locus of their common point.

Let $x'y'$ be a point of the parabola whose equation is $y^2 = 2px$. We have always the subnormal $= p \therefore p + x'$ is the abscissa of the point of intersection of the normal with the axis of x (or the principal axis of the parabola), and the equation of the line drawn

through this point, perpendicular to the normal, or parallel to the tangent, is, therefore,

$$y = \frac{p}{y'} [x - (p + x')].$$

But $by = p(x + a)$ is the equation of the line which joins the points of contact of the two tangents to the parabola which meet in the point a, b . The identity of these equations gives $\frac{p}{y'} = \frac{p}{b}$, or $y' = b$; also, $-\frac{p(p+x')}{y'} = \frac{pa}{b}$, or $p + x' = -a$.

Substituting for x' its value $\frac{y'^2}{2p} = \frac{b^2}{2p}$, there results

$$b^2 = -2pa - 2p^2.$$

The locus is therefore another and equal parabola opening in the opposite direction, with its vertex at the distance p from that of the other.

Either parabola is the locus of the points a, b , in respect to the other parabola.

I find the same reasoning which led me to the solution of this problem, also very readily solves Problem V., No. XI., Vol. I.: — “If through any point, secants be drawn to a curve of the second order, and through the points where each secant intersects the curve, tangents be drawn meeting each other, the locus of the points of meeting is a straight line.”

The equation of a line passing through the given point c, d is $y - c = A(x - d)$. The equation of the line which joins the point of contact of the two tangents to a conic section which meets in the point a, b is $2by = (m + 2an)x + am$. The identity of these equations requires that $\frac{m+2an}{2b} = A$, and $\frac{am}{2b} = c - Ad$,

$$\therefore 2bc = (m + 2an)d + am;$$

and thus
$$b = \frac{2nd + m}{2c} \cdot a + \frac{md}{2c}.$$

The locus of the points a, b is a straight line easily constructed.

PRIZE SOLUTION OF PROBLEM IV.

By ASHER B. EVANS, Madison University, Hamilton, N. Y.

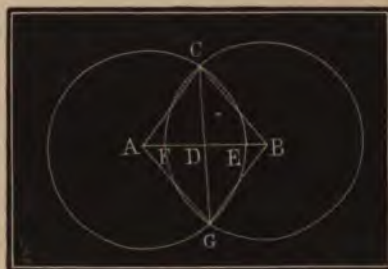
IV. Required the radii of three equal circles tangent to each other, and cutting off equal areas from a given circle, so that the sum of these areas may be a maximum.

Let B represent the centre and BC the radius of the given circle, A the centre and AC the radius of one of the three equal tangent circles. Let $BC = 1$, $AC = r$, area of $FCEG = u$; then will
 $AB = \frac{2}{3} r \sqrt{3}$, $AD = \frac{7r^2 - d}{4r\sqrt{3}}$,

$$BD = \frac{r^2 + 3}{4r\sqrt{3}}, \quad CD = \sqrt{\frac{42r^2 - r^4 - 9}{48r^2}},$$

$$CF = \cos^{-1} \left(\frac{r^2 + 3}{4r\sqrt{3}} \right),$$

$$CE = r \cos^{-1} \left(\frac{7r^2 - 3}{4r^2\sqrt{3}} \right).$$



Area, $FCEG = ACEG + BCFG - ACBG$, or

$$(1) \quad u = r^2 \cos^{-1} \left(\frac{7r^2 - 3}{4r^2\sqrt{3}} \right) + \cos^{-1} \left(\frac{r^2 + 3}{4r^2\sqrt{3}} \right) - \frac{1}{6} \sqrt{(42r^2 - r^4 - 9)}.$$

Differentiating (1) gives

$$(2) \quad \frac{du}{dr} = 2r \cos^{-1} \left(\frac{7r^2 - 3}{4r^2\sqrt{3}} \right) - \frac{1}{3r} \sqrt{(42r^2 - r^4 - 9)}.$$

Putting $\frac{du}{dr} = 0$ gives

$$(3) \quad \cos^{-1} \left(\frac{7r^2 - 3}{4r^2\sqrt{3}} \right) = \frac{2}{\sqrt{3}} \sqrt{1 - \left(\frac{7r^2 - 3}{4r^2\sqrt{3}} \right)^2}.$$

For brevity, put $\frac{7r^2 - 3}{4r^2\sqrt{3}} = x$; then (3) becomes

$$(4) \quad \cos^{-1} x = \frac{2}{\sqrt{3}} \sqrt{(1 - x^2)}.$$

This equation, solved by approximation, gives $x = .60976$,
 $\therefore r = 1.0385$ nearly.

Equation (2) gives

$$\frac{d^2 u}{d r^2} = 2 \cos^{-1} \left(\frac{7 r^2 - 3}{4 r^2 \sqrt{3}} \right) - \frac{12 + \frac{8}{r^2} - \frac{1}{3} r^2}{\sqrt{(42 r^2 - r^4 - 9)}},$$

which is essentially negative when $r = 1.0385$; hence r is the radius of one of the three equal circles that cut off the greatest areas from the given circle.

No correct solution of Problem V. has been received.



REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. X., Vol. II.

The first prize is not awarded.

The second prize is awarded to A. G. BARKER, Waterville College, Maine.

The third prize is awarded to WILLIAM TIMPSON, White Plains, N. Y.

PRIZE SOLUTION OF PROBLEM I.

By A. G. BARKER, Waterville College, Maine.

I. Solve the equations

$$x^2 y^2 (x^4 - y^4) = 2340, \quad x y (x y^3 - 1) (x^2 + y^2) = 1794,$$

by quadratics.

Performing the multiplications indicated, and adding twice the second equation to the first, there results

$$x^6 y^2 + 2 x^4 y^4 + x^2 y^6 - 2 x^3 y - 2 x y^3 = 5928;$$

or,
$$x^2 y^2 (x^2 + y^2)^2 - 2 x y (x^2 + y^2) = 5928.$$

Solving this quadratic, we get

$$x y (x^2 + y^2) = 78, \quad \text{or} \quad -76.$$

Combining this result with the given equations, it is easy to find $x = 3$, $y = 2$. These were the only values of x and y found in any of the solutions.

PRIZE SOLUTION OF PROBLEM II.

By WILLIAM TIMPSON, White Plains, N. Y.

II. Given the lengths of the three perpendiculars dropped from any point in the plane of an equilateral triangle upon the sides; to find the segments of the sides.

Put $DE = a$, $DF = b$, and $DG = c$. Now,

$$\frac{1}{2} AB (DE + DF + DG) = \text{area } \triangle ABC = \frac{1}{2} AB \cdot CH;$$

$$\therefore CH = DE + DF + DG = a + b + c.$$

Making $AH = x$, $(2x)^2 - x^2 = (a + b + c)^2$, or $2x = \frac{2}{\sqrt{3}} (a + b + c)$, the side of the triangle.

The triangles DKL , CFE , CHB are similar; and since $HB = \frac{1}{2} BC$, $FL = \frac{1}{2} LC$, and $LK = \frac{1}{2} DL$. Let $FL = y$, then $CL = 2y$, $DL = b - y$,

$$LK = \frac{b - y}{2} = a + b - 2y \therefore y = \frac{2a + b}{3},$$

$$\text{and } CL = 2y = \frac{2}{3} (2a + b).$$



In the same way, $NG = \frac{2a + c}{3}$, and $AN = \frac{2}{3} (2a + c)$; $PF = \frac{2c + b}{3}$,

and $PB = \frac{2}{3} (2c + b)$. Now $FC = \sqrt{CL^2 - FL^2} = \frac{2a + b}{\sqrt{3}}$,

$$\therefore BF = \frac{2}{\sqrt{3}} (a + b + c) - \frac{2a + b}{\sqrt{3}} = \frac{b + 2c}{\sqrt{3}}.$$

Also $AG = \sqrt{AN^2 - NG^2} = \frac{2a + c}{\sqrt{3}}$, $\therefore GB = \frac{2b + c}{\sqrt{3}}$.

And

$$AE^2 + ED^2 = DG^2 + AG^2,$$

$$\therefore AE = \frac{a + 2c}{\sqrt{3}}, \quad \text{and} \quad CE = \frac{a + 2b}{\sqrt{3}}.$$

we have obtained the following:—

$$\begin{aligned}AE^2 + ED^2 &= AE^2 + ED^2, \\BE^2 + EG^2 &= BG^2 + GD^2, \\CE^2 &= CF^2 + FD^2, \\AE^2 + BG^2 + CF^2 &= AE^2 + BG^2 + CF^2; \\CF^2 + CE^2 - AE^2 &= 0;\end{aligned}$$

$$CF^2 - CF^2 = (CE + AE)(CE - AE) = 0$$

$$CF^2 - CF^2 = CA(CE - AE) = 0.$$

$$CF^2 = CA.$$

$$CF = CE - AE = 0,$$

$$CF = CF + AE;$$

∴ segments of the sides is equal to
the square of the radius.

PROBLEM III.

Let $\theta = 36^\circ$ and $\phi = 36^\circ$, prove that

$$\frac{1}{\sin \theta} = \frac{1}{\sin \phi} + \frac{1}{\sin 2\theta}.$$

Solution.

Let $\theta = 36^\circ$ and $\phi = 36^\circ$, then $2\theta = 72^\circ$.

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Let $\theta = 36^\circ$ and $\phi = 36^\circ$, then $2\theta = 72^\circ$.

Q.E.D.

Let H = area of triangle AEB , and α = angle ACE . Then the angle between the planes ACE and ACB will be $\frac{\pi}{n}$; and that between the planes ACB and BCE will be $\frac{\pi}{m}$; and the intersections of these three planes with the inscribed sphere will form a right spherical triangle, of which one side will be $R\alpha$, and the two angles will be $\frac{\pi}{n}$ and $\frac{\pi}{m}$.

Whence
$$\cos \alpha = \frac{\cos \frac{\pi}{m}}{\sin \frac{\pi}{n}}.$$

In the right triangle AEB , the angle at A will be $\frac{\pi}{n}$, and $EB = \frac{1}{2}e$; whence

$$AE = r = \frac{1}{2}e \cot \frac{\pi}{n}, \quad H = \frac{1}{2}e^2 \cot \frac{\pi}{n},$$

$$G = 2nH = \frac{1}{2}e^2 n \cot \frac{\pi}{n},$$

$$R = r \cot \alpha = \frac{\frac{1}{2}e \cot \frac{\pi}{n}}{\sqrt{\left(\frac{\sin^2 \frac{\pi}{n}}{\cos^2 \frac{\pi}{m}} - 1\right)}},$$

$$V = F \times G \times \frac{1}{2}R = \frac{e^3 F n \cot^2 \frac{\pi}{n}}{24 \sqrt{\left(\frac{\sin^2 \frac{\pi}{n}}{\cos^2 \frac{\pi}{m}} - 1\right)}}.$$

For the

Tetrahedron, $F=4, n=3, m=3, V = \frac{e^3 \sqrt{2}}{12}.$

Cube, $F=6, n=4, m=3, V = e^3.$

Octahedron, $F=8, n=3, m=4, V = \frac{e^3 \sqrt{2}}{3}.$

Dodecahedron, $F=12, n=5, m=3, V = \frac{5e^3 \cot^2 36^\circ}{2\sqrt{(4\sin^2 36^\circ - 1)}} \\ = 5e^3 \sqrt{\left(\frac{47+21\sqrt{5}}{40}\right)} = \frac{5e^3 \tan 45^\circ}{4\sqrt{(\sin 6^\circ \cos 24^\circ)}}.$

Icosahedron, $F=20, n=3, m=5, V = \frac{5e^3 \cot^2 60^\circ}{2\sqrt{\left(\frac{\sin^2 60^\circ}{\cos^2 36^\circ} - 1\right)}} = 5e^3 \left(\frac{3+\sqrt{5}}{12}\right).$

Mr. EVANS's solution of Problem IV. is vitiated by a slight oversight in the last case.

No correct solution of Problem V. has been received, probably owing to a misapprehension of the problem. The bisection referred to in the problem is general, and independent of any law of variation in density. Hence a plane cannot be the bisecting surface.

SIMON NEWCOMB,
W. P. G. BARTLETT,
TRUMAN HENRY SAFFORD.

DISCUSSION OF THE EQUATIONS WHICH DETERMINE
THE POSITION OF A COMET OR OTHER PLANETARY
BODY FROM THREE OBSERVATIONS.

By G. W. HILL, Nyack Turnpike, N. Y.

THESE equations may be found in the *Mécanique Céleste*, Vol. I, p. 207, or in PONTECOULANT, *Théorie Analytique du Système du Monde*, Vol. II, p. 21, where the analysis conducting to them is given. The equations are

$$\varrho = \frac{R^4}{m} \left(\frac{1}{r^3} - \frac{1}{R^3} \right), \quad r^2 = R^2 - 2 R \varrho \cos c + \varrho^2;$$

in which ϱ is the comet's distance from the earth, r its radius vector, R the earth's radius vector, c the angular distance of the comet from the sun, and m a function of known quantities. The unknown quantities are ϱ and r . Now, to find the values of ϱ and r , it has always been advised to determine them from the equations in the above form by the method of double position. But this way is needlessly complex in calculation, and we shall give the equations a much simpler form, to which the tentative process can be easily applied. Assume $r = \frac{R \sin c}{\sin \theta}$, and $\varrho = \frac{R \sin (\theta - c)}{\sin \theta}$; these values substi-

tuted in the last equation render it identical ; and from the first we obtain

$$\frac{\sin^3 \theta}{\sin^3 c} = 1 + m \frac{\sin(\theta - c)}{\sin \theta} ;$$

or, $\sin^4 \theta = \sin^3 c [(1 + m \cos c) \sin \theta - m \sin c \cos \theta]$.

Let $\sin^3 c (1 + m \cos c) = A \cos B$, and $m \sin^4 c = A \sin B$; whence

$$\tan \beta = \frac{m \sin c}{1 + m \cos c}, \quad \text{and} \quad A = \frac{m \sin^4 c}{\sin \beta} = \frac{\sin^4 c}{\sin(c - \beta)}.$$

Then the equation becomes

$$\sin^4 \theta = A \sin(\theta - \beta).^*$$

This is probably the most elegant form to which the above equations can be reduced. Taking the logarithms of both sides, we have

$$4 \log \sin \theta - \log \sin(\theta - \beta) = \log A.$$

From this a near approximate value of θ can be found by inspection of the table of logarithmic sines, without any calculation but that which can be performed mentally. The exact value may then be obtained by a single application of the method of double position.

We proceed to notice some properties of the roots of the equation to which we have reduced the two first given.

The equation may be put under this form :

$$\sin^8 \theta - 2 A \cos \beta \sin^6 \theta + A^2 \sin^4 \theta - A^2 \sin^2 \beta = 0.$$

From the powers of $\sin \theta$, which are wanting, we perceive that the equation has at least four imaginary roots ; and the sign of the last term being negative, there are at least two real roots, one positive, the other negative. By reference to the equations which determine φ and r , it will be seen that they are satisfied by the values $\varphi = 0$,

* See GAUSS'S *Theoria Motus*, and especially the Appendix to COMMANDER DAVIS'S translation. — ED.

and $r = R$; the corresponding value of θ is c , which, put in the equation $\sin^4 \theta = A \sin(\theta - \beta)$, renders it identical. Since, by the nature of the problem only positive values of ρ and r are admissible, and consequently θ being contained between the limits c and π , we may reject as useless the four imaginary roots, and the negative root and the root $\theta = c$. There remain two roots which are necessarily real, because the problem must have at least one solution, and we are led to the important conclusion that it cannot have more than two.

Let us here deduce a relation which exists between the known quantities when these two roots are real. Differentiating the equation $4 \log \sin \theta - \log \sin(\theta - \beta) = \log A$ with respect to θ , we get for the equation containing the values of θ , when

$$4 \log \sin \theta - \log \sin(\theta - \beta)$$

is a maximum or minimum,

$$4 \cot \theta - \cot(\theta - \beta) = 0, \quad \text{or} \quad 4 \tan(\theta - \beta) = \tan \theta;$$

from which we derive

$$\tan \theta = \frac{3 \pm \sqrt{9 - 16 \tan^2 \beta}}{2 \tan \beta}.$$

If $\tan^2 \beta > \frac{9}{16}$, this value of $\tan \theta$ is imaginary, and the expression $4 \log \sin \theta - \log \sin(\theta - \beta)$ is not susceptible of a maximum or minimum value, and the equation in $\sin \theta$ has only two real roots, which are among those rejected. Hence we conclude, that when the data are taken from observation, the quantity $\frac{m \sin c}{1 + m \cos c}$ will always be contained between the limits $\pm \frac{3}{4}$. If we substitute 0 for θ in the equation $\sin^4 \theta - A \sin(\theta - \beta) = 0$, the result is $A \sin \beta$, and for $\theta = \pi$ the result is $-A \sin \beta$; showing the existence of an odd number of roots between the limits $\theta = 0$ and $\theta = \pi$, which odd number is three, since c and the root which the

problem demands are within these limits. If we make $\theta = c + dc$, there results the quantity $(4 \sin^3 c \cos c - A \cos(c - \beta)) dc$; or,

$$\text{since} \quad A = \frac{\sin^4 c}{\sin(c - \beta)}, \quad \tan \beta = \frac{m \sin c}{1 + m \cos c},$$

the quantity $(3 \cos c - m) \sin^3 c dc$. And $-A \sin \beta$, the result on putting $\theta = \pi$, is equal to $-m \sin^4 c$. Therefore, $\sin c$ being always positive, θ has two real values, or only one (between the limits c and π), and consequently the problem two or one answer, according as m and $m - 3 \cos c$ have the same or different signs.

It is evident that $A \cos \beta$ must be positive, in order that the equation in $\sin \theta$ may have three real positive roots; so the quantity $1 + m \cos c$ is always positive, and $\tan \beta$ has the same sign as m . If β be taken between the limits $\pm \frac{\pi}{2}$, A is always positive. Since the equation in $\sin \theta$ must have no root greater than one, units substituted for $\sin \theta$ in the first derived function of its equation must render it positive; that is, the expression $4 - 5 A \cos \beta + A^2$ is positive, which gives $A < 2$, and $A \cos \beta < \frac{4}{5}$. According as m is positive or negative, the equation for finding θ presents itself under two shapes, $\sin^4 \theta = A \sin(\theta - \beta)$, and $\sin^4 \theta = A \sin(\theta + \beta)$, in which A is always positive and less than 2, and β never exceeds $36^\circ 53'$.

From the expression for ρ in terms of r , it is clear that r is less or greater than R , according as m is positive or negative. Therefore, in the first case, θ is contained between the limits c and $\pi - c$; and in the second case, if c is in the first quadrant, between $\pi - c$ and $\pi - \beta$; but if c be in the second quadrant, between c and $\pi - \beta$. These remarks may be of use to shorten the tentative process of finding θ .

With regard to θ , it is clear it is the angle subtended at the comet by its radius vector and the line joining it and the earth prolonged beyond the comet.

Mathematical Monthly Notices.

Elements of Chemical Physics. By JOSIAH P. COOKE, JR., Erving Professor of Chemistry and Mineralogy, in Harvard University. Boston: Little, Brown, & Co. 1860. 8vo. pp. 739.

This is the first of an expected series of three volumes, to form "an extended work on the Philosophy of Chemistry." It is, however, in itself a complete treatise of the principles involved in the processes of weighing and measuring.

The introduction gives a view of the nature of physical and chemical laws as seen by the light of a metaphysical theory of causation, rather doubtful in itself, but very popular among modern theologians.

In the second chapter, under the head of "General Properties of Matter," we have an account of the English and French measures of length, and the elementary principles of mechanics, including some formulæ for the motion of the pendulum. From some of the reasoning and conclusions respecting *vis-viva* in this chapter, we are disposed to dissent, as being inaccurate, and the term "centre of gravity" seems to be sometimes used in a very unusual sense.

Chapter III., entitled "The Three States of Matter," gives a highly interesting, extended, and luminous account of the characteristic properties of solids, liquids, and gases, with many valuable tables. Upwards of sixty pages of this chapter are devoted to crystallography, profusely illustrated by diagrams on a black ground; and eighty pages to the mutual molecular actions between matter in its three forms, taken two and two: e. g. between solids and solids, solids and liquids, etc.

The fourth chapter is devoted to the *chemical* theory of heat; and seems to be much more full and satisfactory than the chapters on that subject in any work previously published in this country. It includes accounts of the results of REGNAULT and DULONG on the specific heat and expansion of gases, of PERSON'S theory of the absolute zero of heat, and of JOULE'S experiments on the mechanical theory of heat, so interesting to the physical philosophers; all the results being deduced mathematically, where necessary, and expressed quantitatively, where it is possible. It contains a vast and patiently collected mass of information.

As a general text-book for the student, it labors under the disadvantage of not completely filling the course of either physics or chemistry, as usually taught in colleges; but we think the third and fourth chapters might with some advantage, and a great deal of pleasure to the student, be substituted for so much of the course of chemistry as treats of the properties of the -yles, -ines, -ones, etc. The style of these chapters is so free, the illustrations so excellent and numerous, and the typography so good, that no one possessed of the slightest taste for physical science could fail to be interested by them. To the special student in chemistry the work must be both interesting and valuable, from its exhaustive character.

Elements of Analytical Geometry and of the Differential and Integral Calculus. By CHARLES DAVIES, LL.D., Professor of Higher Mathematics, Columbia College. 2 vols. in one. New York: A. S. Barnes and Burr. 1860. pp. 204, 194.

That there is a growing interest in the subjects treated in these volumes there can be no doubt. Their study has been gradually working its way into seminaries, academies, and even into many high-schools. In these institutions a large proportion of students end their course of study, and for such the work before us is mainly intended. In view of such a use, the manner of presenting and the arrangement of the propositions, and especially what and how much to retain out of such abundance, become very serious questions. In these respects,

as well as in the clearness and simplicity with which the work is written, the author has been remarkably successful. The discussion of the general equation of the second degree is brief, but very clear and satisfactory. The arrangement is as follows:— Introduction. Book I. Points and Lines in a Plane. Books II. – V. Of the Circle, Ellipse, Parabola, Hyperbola. Book VI. Space, Point, and Line, Surfaces of the First and Second Orders. It is enough to say of this part of the volume, that it is a judicious abridgment of the Author's Analytical Geometry.

But Part Second, to which we wish to call more particular attention, shows even more marked evidence of thought and care in arrangement and detail. With the arrangement we are especially pleased, and believe it will receive the approbation of all practical teachers. In Section I. we find Definitions and First Principles; II. Differentials of Algebraic Functions; III. Integration and Applications; IV. Successive Differentials, &c.; V. Maxima and Minima; VI. Differentials of Transcendental Functions; VII. Transcendental Curves; and, lastly, Integration of Rational Fractions, Integration by Parts, and a Discussion of the Cases in which Binomials can be integrated. It will be seen that the processes of differentiation and integration are combined, instead of the usual division. The integration of the various classes of functions immediately follows their differentiation, and then the related application of both processes. The advantages of this combination of the two converse processes are obvious; and especially when the student has but little time to devote to the subject.

We sincerely regret that we cannot give the same hearty assent to the author's exposition of the first principles of the subject. For brevity, and to be sure of presenting his theory completely, we quote, retaining his italics. "Where no law of change has been fixed, such a law may be imposed as will make the variable to change uniformly, and to pass through all values between any two limits. There exists a constant difference between two consecutive values, which forms no appreciable part of the quantity itself."— *Introduction*.

"If the change is *uniform*, and such as to embrace *every value* between given limits, such change cannot be expressed by a number whose unit is any part of 1. For, however small the decimal denoting the change, it might still be diminished by prefixing ciphers; hence, contrary to the supposition, there would be values which *h* did not pass through. Thus we cannot express the change by numbers, and a new language becomes necessary." § II.

"Before *N* can coincide with *M*, there must be a *last value* of $x' - x$; we denote it by dx ; and $x, x + dx$, are called *consecutive*. *Consecutive values admit of no intermediate value under the same law of change*; hence their difference cannot be expressed by a number, but is, *numerically* = 0. It is named a *differential*." § 13.

" dx is smaller than any number; its numerical value is 0. $(dx)^2$ is 0, compared with dx ; i. e. infinitely less than dx ." § 20.

We infer (1) that *before* $x' - x$ *vanishes*, it has a "last value" which really exists, but which is too small to be further diminished by prefixing ciphers; and that all so-called continuous change is actually composed of an infinity of such steps; (2) that while dx is such a *minimum possible*, yet dy is smaller if the derivative $\frac{dy}{dx}$ be between -1 and $+1$; and that $(dx)^2$ is infinitely less than dx . This is the meaning, unless the proviso "under the same law of change" modifies it in some way which we do not clearly understand, and which may well perplex the young student for whom the book is written. Now (1) is manifestly untrue; it is also incompatible with (2). But were both (1) and (2) true, we think two objections to the theory would remain. First, it involves the mystery of continuity, and also the inconceivable idea of *absolute* infinitesimals or infinites of different orders; while the calculus requires us to discuss neither. Second, it treats differentials as magnitudes *essentially fixed*, while the spirit of the Calculus is to regard them as *essentially arbitrary*. We prefer to define them as any changes

taken so small that the character of the result could not be changed by taking them still smaller; and derivatives as the *limits* toward which $\frac{dy}{dx}$, &c. finally *approach*, as dx approaches zero.

The author demonstrates the theorems of MACLAURIN and TAYLOR separately, and gives no hint that the two are identical. But by making $a = 0$, TAYLOR's development of $f(a+x)$ in powers of x , becomes MACLAURIN's development of $f(x)$; while putting $\phi(x) = f(a+x)$, MACLAURIN's development of $\phi(x)$ becomes TAYLOR's development of $f(a+x)$.

We have thus, as fully as space will permit, pointed out what we consider as defects of, in other respects, an admirable work; one having too many positive merits not to be improved by just adverse criticism.

Elements of Geometry, with Practical Applications to Mensuration. By BENJAMIN GREENLEAF, A. M., Author of "The National Arithmetic," "Treatise on Algebra," &c. Fourth Electrotype Edition. Boston: Published by Robert S. Davis & Co. 1859.

This work is the last of the series yet published, beginning with the "National Arithmetic," which was first issued in 1835. We understand that a work on Plane and Spherical Trigonometry is soon to be added. The "National Arithmetic," by frequent revisions, has been kept fully up with the times. The last revision, in 1857, has left it an excellent work, especially for fitting students for business life, and its clearly expressed rules will always make it a good book for reference, as well as text-book.

The Geometry is based on that of LEGENDRE, and for the sake of comparison we append the table of Contents. Book I. First Principles. II. Ratio and Proportion. III. The Circle, and the Measure of Angles. IV. Proportions, Areas, and Similarity of Figures. V. Problems Relating to the preceding Books. VI. Regular Polygons, Area of Circles. VII. Planes, Dihedral and Polyhedral Angles. VIII. Polyhedrons. IX. The Sphere and its Properties. X. The Three Round Bodies. XI. Applications of Geometry to the Mensuration of Plane Figures. XII. Applications to Solids. XIII. Miscellaneous Geometrical Exercises. XIV. Application of Algebra to Geometry. This will show to what extent the excellent work of LEGENDRE has been reproduced in the one before us. In the different books we notice many changes in the order of the propositions, and a few in the methods of demonstration; and many of them appear to us as the result of experience in the school-room, and as better adapting the work to the wants of beginners. But in our opinion its crowning excellence is the clearness and conciseness with which the definitions and demonstrations are expressed. The greatest care has evidently been taken with the style; and after a careful search we have not been able to find one expression which we felt sure could be improved. Book XIII, comprising undemonstrated propositions, and problems for construction and numerical solution, is another valuable feature, and the only fault is that they are too few. For instance, only ten of the undemonstrated propositions relate to Book I., when there ought to have been at least thirty-five, one to each of the demonstrated propositions. Besides, these propositions and problems would be more likely to be used if they had been inserted in their proper places throughout the book, just what should have been done with Books XI. – XIX. A book on Arithmetic or Algebra would not be tolerated with so few exercises, and all put at the end. There are advantages in not bringing in the definitions before they are wanted. If the definitions of parallels had been based on the idea of direction, as that of an angle is, the change from LEGENDRE would, in our opinion, have simplified several of the earlier propositions much more than that adopted by the author has done. The changes we have suggested are matters of opinion, and by no means imply that the work is not one which we can heartily recommend to the attention of teachers. Indeed, we believe that teachers will generally introduce all the exercises and applications in their proper places, and thus remove our only objection to the arrangement.

Supplément au traité de

Mécanique céleste

pour l'impression du traité de mécanique céleste, j'ai publié
les divers objets de cet ouvrage, un grand nombre de recherches
verifiées dans les mémoires, de l'Institut et de l'Académie des Sciences
et dans les volumes de la Correspondance de Paris. Je vais ici réunir
ces recherches dans l'ordre qui me paraît le plus convenable, et
en leur donnant toute la généralité dont elles méritent par leur portée.
Je vais commencer par ce qui concerne la figure des Corps céles-
tes, et les oscillations des fluides qui les recouvrent.

Autographe de Laplace.

Cette note, écrite de la main du célèbre géomètre français a été donnée à M. Watson
par Madame la Marquise de Laplace de veuve, qui en a certifié en ma présence la
parfaite authenticité.

Ch. De launay

THE MATHEMATICAL MONTHLY.

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PRIZE PROBLEMS FOR STUDENTS.

I. PROVE that the surface of a sphere can be completely covered with the surfaces either of 4, or of 8, or of 20 equilateral spherical triangles.

II. Which is greater, $\frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \&c.$, *ad inf.*, or $\frac{1}{8} + \frac{1}{9} + \frac{1}{27} + \&c.$, *ad inf.*?

III.
$$\frac{(x+a+b)^s + (x+c+d)^s}{(x+a+c)^s + (x+b+d)^s} = \frac{m}{n}.$$

Find x by quadratics.

IV. A fraudulent tradesman contrives to employ his *false* balance both in buying and selling a certain article, thereby gaining at the rate of 11 per cent more on his outlay than he would gain if the balance were *true*. If however the scale-pans, in which the article is weighed when bought and sold respectively, were interchanged, he would neither gain nor lose by the article. Find the legitimate gain per cent on the article.

V. Show that in the parabola the half sum of the radii vectores drawn to the extremities of any arc whatever, is equal to the radius vector drawn to the summit of the diameter passing through the centre of the chord and parallel to the axis, plus the part of this diameter intercepted between the arc and the chord.—Communicated by ASAPH HALL, Esq.

Solutions of these problems must be received by January 1, 1860.

REMARKS ON THE PROBLEM OF THE LIGHTS.*

By PROF. W. H. PARKER, Middlebury College, Vermont.

I AM very much obliged to you for showing, in your editorial remarks on my "Note" in the February number, page 160, Vol. I, that the error which I conceived to be one of interpretation, really consists in a fallacy, to which the misinterpretation is only incidental.

Having re-examined the subject from a new point of view, suggested by your remarks, and with particular reference to the question, how can we determine from our equations whether the conditions of a problem are impossible or not? I desire, with your permission, to add the following remarks.

The general equation of our problem is

$$(A) \quad \frac{a}{x^2} = \frac{b}{(c-x)^2},$$

whose roots are $\frac{c\sqrt{a}}{\sqrt{a} \pm \sqrt{b}}$. If in the roots we suppose $c=0$, both values of $x=0$. In order to determine whether we have here the true roots of our equation, under the particular supposition, let us go back to equation (A), and change it to the form which will express the condition $c=0$. We have now

$$(B) \quad \frac{a}{x^2} = \frac{b}{x^2},$$

the same form which we should have obtained by stating the particular problem which this enunciates. Our roots 0 do not satisfy this equation. A moment's consideration is sufficient to show why they should not. So long as a and b are unequal, the expression

* These Remarks reply to some incidental points in our article, the correctness of the main question being admitted. It is proper to state that this article was received shortly after the publication to which it refers, and has been delayed for want of space. — Ed.

$\frac{a}{x^2} = \frac{b}{x^2}$ is not a true equation; for we have unequals a and b divided by the same quantity, x^2 , and no value of x (not even $x=0$) will make the members equal. This shows conclusively the impossibility of the conditions. When, therefore, our problem requires us to find the point or points of equal illumination of two unequal lights, which are placed together, it requires what the algebraic enunciation of the conditions shows to be impossible.

In like manner, if we take the form of the problem in the editorial remarks, whose equation is

$$(C) \quad \frac{a}{(x-a)^2} = \frac{b}{(\beta-x)^2},$$

and whose roots are $\frac{\beta\sqrt{a} \pm \alpha\sqrt{b}}{\sqrt{a} \pm \sqrt{b}}$; and in the roots suppose $\beta = \alpha$, we obtain $x = \alpha$. To determine whether α is a true root, let us go back to equation (C), from which it was derived. Making $\beta = \alpha$, we get

$$(D) \quad \frac{a}{(x-a)^2} = \frac{b}{(\alpha-x)^2},$$

a form of equation which α will not satisfy. Hence α is not one of the roots. Indeed the expression $\frac{a}{(x-a)^2} = \frac{b}{(\alpha-x)^2}$ has no roots, for the reason that it is not an equation. For here, too, we have unequals a and b divided by equals; and here, just as in the former case, we find the conditions to be impossible, because we see that the expressions for the intensities cannot be equal.

With the knowledge that the expressions $\frac{a}{x^2} = \frac{b}{x^2}$ and $\frac{a}{(x-a)^2} = \frac{b}{(\alpha-x)^2}$ are not equations at all, it is absurd for us to attempt their reduction, or to ask for their roots. From false premises, surely, we cannot expect reliable conclusions.

I deem it important to remark here, that, since 0 is not a true root of equation (B), nor α of equation (D), we have in neither of them the symbol of infinity. How, then, is the impossibility of the conditions made known? Simply, as it seems to me, by the untruth-

fulness of the equations ; or, in other words, by the impossibility of forming the conditions into real equations.

The idea that an impossibility in the conditions will always manifest itself somewhere by the symbol of infinity, I cannot think to be well founded. In the cases already considered, we have an impossibility without the symbol of infinity. On the other hand, we may have the symbol, in a true equation, without any impossibility in the conditions. This is made apparent by considering the case in which $a = b$ and $c = 0$ in equation (A), producing the form $\frac{a}{x^2} = \frac{a}{x^2}$. This is an identical equation, and of course true for any value of x , including $x = 0$. When $x = 0$, we have in each member the symbol of infinity, though we have no impossibility in the conditions. For when the lights are together, and of equal intensity, the point where they are placed is truly a point of equal illumination.

If in equation (C) we make $a = b$ and $\beta = \alpha$, it comes $\frac{a}{(x-a)^2} = \frac{a}{(\alpha-x)^2}$, which is also an identical equation, and, as before, a case of real indetermination. The equation is true for any value of x , including $x = \alpha$. But when $x = \alpha$, each member is a symbol of infinity ; and yet here we have no impossibility, because when the equal lights are placed together at the distance α from the origin, the point they occupy is equally illuminated by them.

It will be observed that I have not been speaking of the symbol of infinity in the *roots* of an equation.

The error of BOURDON seems to me to have originated in his failing to observe that the supposition $c = 0$, in the equation of the problem, destroys the equality of the members, and makes his premises false. Reasoning from them as if true, which is his first error, he falls into another at the point indicated in the editorial remarks, page 179 ; not precisely because, in any form of the problem, the particular *hypothesis* introduces a factor *zero* (since α is not a true root of (D), nor 0 of (B)), but because, in reducing the expression

(B) as if it were an equation, he obtains a form $(ax^2 = bx^2$, or $(a-b)x^2 = 0$) whose members would be equal, if $x = 0$, and hence infers that x is equal to 0 in form (B); thus surreptitiously admitting the factor *zero*, rather than legitimately introducing it by the particular hypothesis.

In a true equation, I do not discover that the presence of *zero* as a factor leads to any fallacy, so long as the factor remains. The removal of a factor *zero* from a real equation may introduce a fallacy, but it is one which the algebraist is apt to discover if he has inadvertently fallen into it.

One word in reference to a seeming antagonism between my remarks on the reliability of the algebraic processes as a method of reasoning, and the editorial remarks in regard to the "*implicit fallacy* to which algebraic transformations are often liable." The difference I suppose to be rather apparent than real. I have been accustomed to regard the fallacies which sometimes occur in our use of the algebraic methods as resulting from errors in ourselves. When our first equation is true, and our transformations are legitimately made, i. e. in strict accordance with those necessary truths which underlie all changes of form in equations, I have supposed we might always rely implicitly upon our conclusions. In this confidence I can hardly think myself singular.

If it be true, that there are no fallacies inherent in the algebraic processes, is it not better to speak of the fallacies to which *we* are often liable in algebraic transformations, than to use a form of expression which seems inconsistent with the absolute integrity of those processes as a method of reasoning?

The above remarks are submitted, in the hope that, if they fail to illustrate the topics discussed, they will at least furnish the occasion for remarks from yourself or others, which shall bring the truth to light.

NOTE ON THE PROBLEM OF THE LIGHTS.

By HUGH GODFRAY, St. John's College, Cambridge, England.

MAY I be allowed to make a few remarks on the difficulty in the interpretation of the Problem of the Two Lights, noticed by Professor PARKER in Vol. I, page 160, of the Monthly?

Let us consider the problem somewhat differently, and suppose the two lights together at the outset.

$\overline{D' \quad Q' \quad O \quad P \quad Q \quad D}$

Let two lights, A, B , of intensities a, b , at unit of distance, be placed together at O in the line $D'D$, and let a point, P , which receives light from A , move from D through O towards D' ; we can find the corresponding positions of a point, Q , which moves in such a way as to receive from B the same illumination as P does from A . (We suppose b to have a finite ratio to a .)

The equation connecting the distances will be

$$\frac{a}{OP^2} = \frac{b}{OQ^2};$$

whence $OQ = \pm OP \sqrt{\frac{b}{a}},$

which shows that there are two positions, Q, Q' , at equal distances to the right and left of O corresponding to each position of P .

Let us confine our attention to the $+$ sign, which gives that position of Q which is on the same side of O as P .

$$OQ = OP \sqrt{\frac{b}{a}}.*$$

* If we suppose that OQ and OP are both absolute zeros, that is, that the two lights occupy absolutely the same point, then the equation $OQ = OP \sqrt{\frac{b}{a}}$ is not true, because $\frac{OQ}{OP} = 1$, and not the constant ratio of $\sqrt{\frac{b}{a}}$. A light of a fixed intensity at a unit's distance must illuminate

We see that OQ is in a constant ratio to OP . When OP diminishes, OQ diminishes; when OP becomes negative, or P passes to the left of O , OQ also becomes negative, and Q passes to the left of O , the two points P and Q passing through O simultaneously, a result which agrees with that of the particular case in the usual way of considering the problem; but perhaps the above method of arriving at it may tend to explain away the difficulty experienced in the other.

It is true that when the lights are of unequal intensity there is no point on either side of O equally illuminated; for if OP has any finite value, OQ has a different value; but the distance between these two points, which is

$$OP \frac{\sqrt{b}-\sqrt{a}}{\sqrt{a}},$$

diminishes with OP , and vanishes when OP vanishes, whatever be the relative intensities of A and B .

The value *zero*, therefore, in the usual problem is neither superfluous nor erroneous. The apparent contradiction involved in the statement, that if we admit this value we must also admit that the point O is at the same time equally and unequally illuminated, will disappear if we consider that when the point P is indefinitely near to O , the point Q , which is equally illuminated, is also indefinitely near to O ; and points at which the illumination from B is 2, 3, 4, &c. times that at P will at the same time, as can be shown in a similar manner, be indefinitely near to O , all these points coinciding with O at the same instant that P does.

the point it occupies by a fixed amount, if it illuminates this point at all; and unless we can arbitrarily change the intensity, without producing a corresponding change in the amount by which this point is illuminated, the ordinary solution of the problem for the special case when the two lights are together must be untrue.

We have already shown (page 178, Vol. I.), that, under the hypothesis of the lights occupying the same point, the problem fails altogether, it being impossible to satisfy the corresponding equation with *zero*, or any other value of x . — ED.

and others would therefore be
which I consider a perfectly legit-

... into algebraic language invests
... we introduce admit of, and the
... nearly always more general and
... the particular question we proposed
... of interpretation or superfluous solu-
... have been introduced in the process
... the impossibility of expressing in the
... and limitations of the problem, which is
... instance.



APPROXIMATING TO THE TIME OF DESCENT IN A CIRCULAR ARC.

BY J. V. KEMMER, College of Charleston, South Carolina.

Let AB be a circular arc, whose centre is S , urged by a constant accelerating force, acting always parallel to DB , the radius perpendicular to the circle; it is required to find T , the time of descent, being supposed not more than a quadrant.

From the point from which the descent begins, draw the tangent AD ; on EB the versed sine of the arc of descent AB draw the semi-circle EFB , whose centre is C ; let AC be the chord of the arc AB , and draw the other lines as shown in the figure, whose construction will be easily understood.

Suppose the particle descending from A to have reached the point P , and about to describe the element PQ in the next element of descent with the acquired velocity continued uniformly; put f to express the intensity of the accelerating force, or the additional velocity

acquired in each second of time; also r for the radius SP or SB , or length of the simple pendulum vibrating in the given arc; $\sqrt{(2f \times EO)}$ will express the velocity acquired in descending AP , with which velocity the element PQ is described.

From the elementary triangle PQR , and the similar one SPO , we have

$$PQ \div QR = SP \div PO;$$

from the elementary triangle FCH , and the similar one FEO , we have

$$FH \div FO = FG \div FC;$$

by properties of the circle, we have

$$FO \div \sqrt{(EO \times OB)} = PO \div \sqrt{(DO \times OB)};$$

also write the identity $\sqrt{2f} = \sqrt{2f}$;

and the equality $SP = \sqrt{(\frac{1}{2} DB \times SB)}$.

Hence, observing that $FH = QR$, we obtain

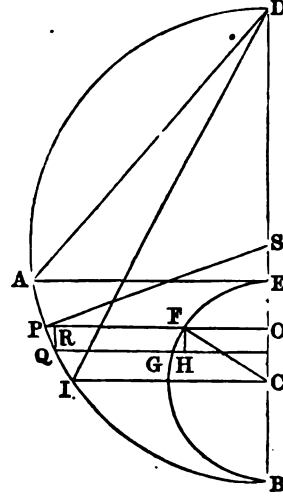
$$\frac{PQ}{\sqrt{(2f \times EO)}} = \frac{FG}{FC} \times \sqrt{\frac{\frac{1}{2} SB}{2f}} \times \sqrt{\frac{DB}{DO}}.$$

This may be written

$$\frac{\text{elementary arc } PQ}{\sqrt{(2f \times EO)}} = \frac{\text{elementary arc } FG}{2FC} \times \sqrt{\frac{r}{f}} \times \sqrt{\frac{DB}{DO}}.$$

Now, since $\frac{\text{elementary arc } PQ}{\sqrt{(2f \times EO)}} = \frac{\text{space}}{\text{velocity}} =$ the element of time in which is described the elementary space PQ , if $\sqrt{\frac{DB}{DO}}$ were a constant, we could write, the sum of the elements of time employed in descending the successive elements of space of which PQ is one =

$$\frac{\text{sum of the elementary circular arcs of which } FG \text{ is one}}{\text{diameter of the circle}} \times \sqrt{\frac{r}{f}} \times \sqrt{\frac{DB}{DO}},$$



and since the sum of the elements of time constitutes T , the whole time of descent down AB , and the sum of the elementary circular arcs, of which FG is one, constitutes the semi-circumference EFB , and $\frac{\text{semi-circumference}}{\text{diameter}}$ is equal to $\frac{\pi}{2}$, we might write

$$T = \frac{\pi}{2} \sqrt{\frac{r}{f}} \times \sqrt{\frac{DB}{DO}}.$$

But $\sqrt{\frac{DB}{DO}}$ has a different value for every such elementary arc as FG , and we require, not the sum of the elementary arcs, but the sum of the products of each elementary arc by the corresponding value of $\sqrt{\frac{DB}{DO}}$; this summation constitutes the difficulty of the problem, and may be effected by the use of arcs of the ellipse and hyperbola. In its place we will substitute an approximation by using a *mean value* of $\sqrt{\frac{DB}{DO}}$.

The greatest value of $\sqrt{\frac{DB}{DO}}$ is $\sqrt{\frac{DB}{DE}}$, and the least is $\sqrt{\frac{DB}{DB}} = 1$, the arithmetical mean of these values is $\frac{1}{2} (1 + \sqrt{\frac{DB}{DE}})$, which, substituted in the equation for T , gives

$$T = \frac{\pi}{2} \sqrt{\frac{r}{f}} \times \frac{1}{2} (1 + \sqrt{\frac{DB}{DE}}).$$

By the properties of the circle, $\sqrt{\frac{DB}{DE}} = \frac{DA}{DE}$, and this ratio is the secant of the angle ADB , or the secant of half the arc AB ; if this arc AB , the arc of excursion of the pendulum, be A , then

$$T = \frac{\pi}{2} \sqrt{\frac{r}{f}} \times \frac{1 + \sec \frac{1}{2} A}{2},$$

as an approximate value for the time. The subjoined table gives, for every 10° of arc A in the first column, the corresponding value of $\frac{1 + \sec \frac{1}{2} A}{2}$, in the second column.

Another mode of approximation presents itself. The greatest

value of DO is DB , and the least is DE , and we may take as its mean value DC , which corresponds to the centre of the circle described on EB as a diameter, and substitution gives

$$T = \frac{\pi}{2} \sqrt{\frac{r}{f}} \times \sqrt{\frac{DB}{DC}};$$

as $\frac{DB}{DC} = \frac{DI}{DO} = \secant \text{ of } IDC$, or the secant of half the arc IB , if this arc be expressed by a , we shall have $T = \frac{\pi}{2} \sqrt{\frac{r}{f}} \times \sec \frac{1}{2} a$. The arc a may be found from A , the given arc of excursion by either of the following equations: —

$$\begin{aligned} \text{versed sine } a &= \frac{1}{2} \text{versed sine } A; & \cos a &= \cos^2 \frac{1}{2} A; \\ \sin \frac{1}{2} a &= \sin \frac{1}{2} A \sin 45^\circ. \end{aligned}$$

If EB , the height of the arc of descent AB , be expressed by h , then $\text{versed sine } a = \frac{h}{2r}$. The table gives, for every 10° of arc A in the first column, the corresponding value of arc a in the third column, and of $\sec \frac{1}{2} a$ in the fourth.

As there is nothing in the investigation to indicate which method will give the closest approximation to the truth, we may take the arithmetical mean of the two as a final result, and we obtain

$$T = \frac{\pi}{2} \sqrt{\frac{r}{f}} \times \frac{1 + \sec \frac{1}{2} A + 2 \sec \frac{1}{2} a}{4},$$

as a final approximate expression for the time of descent in a circular arc of A degrees, A being not more than 90° . The fifth column of the table gives the mean of the second and fourth columns, which corresponds to the last factor in the preceding expression. As neither of the two approximations differs from the mean by half a unit in the fifth decimal place when $A = 10^\circ$, either may be used for cal-

A .	$\frac{1 + \sec \frac{1}{2} A}{2}$	a .	$\sec \frac{1}{2} a$.	Mean of 2d & 4th Columns.
0		0		
10	1.001910	7.32	1.001904	1.001907
20	1.007713	14.06	1.007618	1.007666
30	1.017638	21.05	1.017168	1.017813
40	1.032089	28.00	1.030614	1.031351
50	1.050689	34.47	1.047954	1.049822
60	1.077350	41.25	1.069011	1.073181
70	1.110387	47.51	1.093929	1.102159
80	1.152704	54.04	1.122659	1.137681
90	1.207107	60.00	1.154701	1.180904

culating T when A is less than 10° ; when A is 5° , neither will differ from the mean in the sixth decimal place. The first will be the most convenient expression when A is given; the second when h is given.

The double of either of the three expressions for T will of course be the time of one complete vibration of the pendulum, whose length is r and angle of excursion A . When A is very small, the last factor in each of the above expressions will differ very little from unity, and we may then put $T = \frac{\pi}{2} \sqrt{\frac{r}{f}}$, the double of which is the time of vibration in the arc of a cycloid whose axis is vertical and equal to $\frac{r}{2}$. The geometrical construction used above can easily be applied to the cycloid, to deduce this result.

It may, perhaps, be assumed with safety, that, for small values of A , our final result does not differ from the truth more than the two first approximations do from each other; but, for large values of A , its error can only be determined by comparing it with the true value of T , furnished by the analytic treatment of the problem. As this value is expressed by $\frac{\pi}{2} \sqrt{\frac{r}{f}}$ multiplied by a certain series, we must expand the last factor in each of our expressions above into a series, and then we need only compare the series together, as the quantity $\frac{\pi}{2} \sqrt{\frac{r}{f}}$ is common to all the expressions. Since DB is $2r$, $DE = 2r - h$, $DC = 2r - \frac{1}{2}h$, in the first mode of approximation,

$$\frac{1}{2} \left(1 + \sqrt{\frac{DB}{DE}} \right) = \frac{1}{2} \left(1 + \sqrt{\frac{2r}{2r-h}} \right) = \frac{1}{2} \left(1 + \left(1 - \frac{h}{2r} \right)^{-1} \right),$$

and in the second mode,

$$\frac{DB}{DC} = \sqrt{\frac{2r}{2r-\frac{1}{2}h}} = \left(1 - \frac{1}{2} \cdot \frac{h}{2r} \right)^{-1};$$

developing these expressions by the binomial theorem, we shall have

the first and second series given below (the third line contains the arithmetical mean of the two), which series is rewritten on the fourth line, slightly altered in form for readier comparison with the true series, which is written on the fifth line. In the general term, n indicates the order, beginning with the second.

$$\begin{aligned}
 & 1 + \frac{1}{2} \cdot \frac{1}{2} \left(\frac{h}{2r} \right) + \frac{1}{2} \cdot \frac{1 \cdot 3}{2 \cdot 4} \left(\frac{h}{2r} \right)^2 + \frac{1}{2} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \left(\frac{h}{2r} \right)^3 + \frac{1}{2} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \left(\frac{h}{2r} \right)^4 + \&c. \\
 & 1 + \frac{1}{2} \cdot \frac{1}{2} \left(\frac{h}{2r} \right) + \frac{1}{2^2} \cdot \frac{1 \cdot 3}{2 \cdot 4} \left(\frac{h}{2r} \right)^2 + \frac{1}{2^3} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \left(\frac{h}{2r} \right)^3 + \frac{1}{2^4} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \left(\frac{h}{2r} \right)^4 + \&c. \\
 & 1 + \frac{1}{2} \cdot \frac{1}{2} \left(\frac{h}{2r} \right) + \frac{1}{2} \cdot \frac{2+1}{2^2} \cdot \frac{1 \cdot 3}{2 \cdot 4} \left(\frac{h}{2r} \right)^2 + \frac{1}{2} \cdot \frac{2^2+1}{2^3} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \left(\frac{h}{2r} \right)^3 + \frac{1}{2} \cdot \frac{2^3+1}{2^4} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \left(\frac{h}{2r} \right)^4 + \&c. \\
 & 1 + \frac{1}{2} \cdot \frac{1}{2} \cdot \left(\frac{h}{2r} \right) + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1 \cdot 3}{2 \cdot 4} \left(\frac{h}{2r} \right)^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \left(\frac{h}{2r} \right)^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \left(\frac{h}{2r} \right)^4 + \&c. \\
 & 1 + \frac{1}{2} \cdot \frac{1}{2} \cdot \left(\frac{h}{2r} \right) + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1 \cdot 3}{2 \cdot 4} \cdot \left(\frac{h}{2r} \right)^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \left(\frac{h}{2r} \right)^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \left(\frac{h}{2r} \right)^4 + \&c.
 \end{aligned}$$

On comparing the last two lines, it will be seen that the series furnished by our third or final approximation does not diverge from that furnished by analysis, until the term containing the *fourth power* of the variable $\frac{h}{r}$ the versed sine of A , and errs in excess; yet for $A = 60^\circ$, it gives a result *true to the sixth decimal place*, and, even for $A = 90^\circ$, gives an excess of only half a unit in the third decimal place; that is, of only one five-hundredth of the whole. It is therefore abundantly sufficient for the determination of the variation in the rate of a clock depending on the variation in the extent of the arc of vibration.

The idea of using an approximation to avoid the difficulties of the problem is not new, nor is the geometric construction, adopted in the demonstration, presented as novel, although we are unable to say who first used it; but we have never seen it handled in such a manner as to secure the advantages that belong to the geometric method when the subject is within its scope. Sometimes the calculus is applied to the geometric construction, and then the indepen-

dence and comprehensiveness of the calculus is sacrificed in the combination. We have endeavored so to exhibit the subject, as to secure the simplicity and directness which are the characteristics of the geometric method, and our three modes of approximation, which seem very naturally to present themselves, are, as far as we are aware, new, — never having fallen in our way, — and, with the aid of the usual trigonometrical tables, furnish a far readier means of computing the time in a circular arc than does the series given by the calculus.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE
PRIZE PROBLEMS IN No. XI., Vol. II.

THE first Prize is awarded to ASHER B. EVANS, Madison Univ., N. Y.

The second prize is awarded to JOHN A. WINEBRENER, Princeton College, N. J.

The third prize is awarded to E. O. GIBSON, South New Berlin, N. Y.

PRIZE SOLUTION OF PROBLEM I.

Prove that an equation of the second degree cannot have more than two roots.

The general equation of the second degree is

$$Ax^2 + Bx + C = 0.$$

If this equation can have more than two roots, say three, denote them by a, b, c ; then

$$Aa^2 + Ba + C = 0, \quad Ab^2 + Bb + C = 0, \quad Ac^2 + Bc + C = 0.$$

By subtraction, and dividing by the factors $a - b, a - c, b - c$, which by hypothesis are not zero, we get

$$A(a + b) + B = 0, \quad A(a + c) + B = 0, \quad A(b + c) + B = 0.$$

By subtracting again,

$$A(b-c) = 0, \quad A(a-c) = 0, \quad A(a-b) = 0.$$

But these equations are impossible, since A is not zero. Hence, the given quadratic cannot have three different roots.

PRIZE SOLUTION OF PROBLEM II.

By E. O. GIBSON, South New Berlin, New York.

The sum of n numbers in arithmetical progression is a , and the sum of their squares is b . Required, the first term and the common difference.

Let x denote the first term, and y the common difference; then

$$(1) \quad x + (x+y) + (x+2y) + \dots + (x+(n-1)y) = a.$$

$$(2) \quad x^2 + (x+y)^2 + (x+2y)^2 + \dots + (x+(n-1)y)^2 = b;$$

or, by reduction,

$$(3) \quad nx + \frac{1}{2}n(n-1)y = a,$$

$$(4) \quad nx^2 + n(n-1)xy + \frac{1}{6}n(n-1)(2n-1)y^2 = b.$$

From n times (4) subtract the square of (3); we get

$$\frac{n^2(n^2-1)}{12}y^2 = bn - a^2. \quad \therefore y = \pm \frac{2}{n} \sqrt{\left(\frac{3(bn-a^2)}{n^2-1}\right)}.$$

By substituting this value of y in (3), we find

$$x = \frac{a}{n} \pm \frac{1}{n} \sqrt{\left(\frac{3(n-1)(bn-a^2)}{n+1}\right)}.$$

PRIZE SOLUTION OF PROBLEM III.

Given, three lines, a, b, c , drawn from any point within a regular polygon of n sides to any three of its consecutive corners, A, B, C . Required, the side of the polygon, and also a geometrical solution, granting that a polygon of n sides can be constructed when one side is given.

Upon the line AB , taken equal to (b) , construct the polygon of (n) sides, $AB C D E F$ (in the figure $n = 6$), and join two alternate corners, as A and C , by the diagonal AC . From A as a centre,

$$S = na + \frac{n(n-1)}{1.2} d = \frac{n}{2} (a + a + (n-1)d) = \frac{n(a+b)}{2}.$$

In a geometrical progression,

$$S = a + ar + ar^2 + \dots$$

$$d_1 = a(r-1), \quad d_2 = a(r-1)^2, \quad d_3 = a(r-1)^3, \text{ \&c.}$$

Therefore (1) becomes

$$\begin{aligned} S &= na + \frac{n(n-1)}{1.2} a(r-1) + \frac{n(n-1)(n-2)}{1.2.3} a(r-1)^2 + \text{\&c.} \\ &= \frac{a}{r-1} \left\{ n(r-1) + \frac{n(n-1)}{1.2} (r-1)^2 + \frac{n(n-1)(n-2)}{1.2.3} (r-1)^3 + \text{\&c.} \right\} \\ &= \frac{a}{r-1} \left\{ (r-1+1)^n - 1 \right\} = \frac{a(r^n-1)}{r-1}, \end{aligned}$$

since the series in $\{ \}$ lacks 1 of being the n th power of $1 + (r-1)$.

PRIZE SOLUTION OF PROBLEM V.

By JOHN A. WINEBRENER, Princeton College, New Jersey.

If three points on the surface of a sphere (radius = 1) be joined by three arcs of small circles whose angular radii are φ , φ' , φ'' , the area of the included triangle will be

$$\theta + \theta' + \theta'' + \pi \pm \varphi \cot \varphi \pm \varphi' \cot \varphi' \pm \varphi'' \cot \varphi'';$$

in which φ , φ' , φ'' are the lengths of these arcs, and θ , θ' , θ'' the angles at which they intersect.

Let ABC be the given triangle. Take any point P within it, and join PA , PB , PC . Let O be the pole of BC , or φ , and join OB , OC . Since $\sin \varphi$ is the planar radius of φ , we have by geometry

$$(1) \quad \angle BOC : \varphi = 1 : \sin \varphi. \quad \therefore \angle BOC = \frac{\varphi}{\sin \varphi}$$

The surface of a zone, of which OBC is a part, equals $2\pi(1 - \cos \varphi)$. Therefore the surface of

$$(2) \quad OBC = \frac{\varphi}{\sin \varphi} (1 - \cos \varphi).$$



Denote the angles of the triangle PBC by $\alpha, \alpha', \alpha''$ then the surface of

$$(3) \quad PBOCP = \alpha + \alpha' + \alpha'' - \pi - \frac{\varphi}{\sin \varphi},$$

since $\angle PBC = \alpha' - 90^\circ$, &c.

Adding (2) and (3) the surface of

$$(4) \quad PBC = \alpha + \alpha' + \alpha'' - \pi - \varphi \cot \varphi.$$

The triangles APB and APC will give similar results, and by adding these, and noting that the angles $A + B + C = \theta + \theta' + \theta''$, and the angles at $P = 2\pi$, we have finally the surface of

$$(5) \quad ABC = \theta + \theta' + \theta'' - \pi \pm \varphi \cot \varphi \pm \varphi' \cot \varphi' \pm \varphi'' \cot \varphi''.$$

Note by ASHER B. EVANS.— If $\theta_1, \theta_2, \dots \theta_n; \varphi_1, \varphi_2, \dots \varphi_n; \rho_1, \rho_2, \dots \rho_n$ represent similar quantities for a polygon formed on the surface of a sphere (radius = 1), by joining n points in order, by arcs of small circles, then the area of this polygon will be

$$(6) \quad \theta_1 + \theta_2 + \dots \theta_n - (n-2)\pi \pm \varphi_1 \cot \varphi_1 \pm \varphi_2 \cot \varphi_2 \pm \dots \varphi_n \cot \varphi_n.$$

If $n = 3$, (5) and (6) are identical; if $n = 2$, (6) becomes

$$\theta_1 + \theta_2 \pm \varphi_1 \cot \varphi_1 \pm \varphi_2 \cot \varphi_2 = 2\theta_1 \pm \varphi_1 \cot \varphi_1 \pm \varphi_2 \cot \varphi_2$$

which is the area of a lune formed by the intersection of two small circles, whose radii are ρ_1 and ρ_2 , angle of intersection θ_1 , φ_1 and φ_2 denoting the lengths of the arcs.

SIMON NEWCOMB,

W. P. G. BARTLETT,

TRUMAN HENRY SAFFORD.

NOTES AND QUERIES.

NOTE ON THE GREATEST COMMON MEASURE.

I. The difference between two multiples of a number must be a multiple of that number.

II. Any times a multiple of a number must be a multiple of that number.

Admit the above, and the following solutions are easily understood.

1. Find the greatest common measure of 6 and 28; 72 and 486; 252 and 462; 1573 and 3553; 1181, 2741, and 3413.

2. Mathematical Monthly, Vol. I. p. 5, Ex. 2. "Required the greatest common divisor of 1633, 3763, 4757, and 4189."

SOLUTION OF NO. I.

$$\begin{array}{rclcl}
 6) 28 & 5 & 72) 486 & 7 & 252) 462 & 2 & 1573) 3553 & 2 \\
 \underline{30} & & \underline{504} & & \underline{504} & & \underline{3146} & \\
 2 \text{ Ans.} & & 18 \text{ Ans.} & & 42 \text{ Ans.} & & 407) 1573 & 4 \\
 & & & & & & \underline{1628} & \\
 & & & & & & 55 = 5.11. & \\
 & & & & & & \text{Ans. 11.} &
 \end{array}$$

$$\begin{array}{rcl}
 1181) 2741 & 3413 \\
 \underline{2362} & \underline{3543} \\
 379 & 130 = 10.13. \text{ Neither 2, 5, nor 13 measures 379.} \\
 & \therefore \text{Ans. 1.}
 \end{array}$$

SOLUTION OF NO. II.

$$\begin{array}{rclcl}
 1633) 3763 & 4757 & 4189 & & \\
 \underline{3266} & \underline{4899} & \underline{4899} & & \underline{1633} \\
 497 = 7.71 & 142 = 2.71 & 710 = 10.71 & & 71 = 23. \\
 & & & & \text{Ans. 71.}
 \end{array}$$

The advantage, in many cases, of taking a quotient figure too large by unity, which may be done in any stage of the process, is readily seen. The products could be written over the dividend, if thought necessary.—E. HUNT, Sub-Master English High School, Boston.

NOTES FROM TODHUNTER'S ALGEBRA.

I. *Determine the conditions which make the binomial $a \pm \sqrt{b}$ a perfect cube.*

If we assume

$$(1) \quad \sqrt[3]{a + \sqrt{b}} = x + \sqrt{y}; \text{ then } (2) \quad \sqrt[3]{a - \sqrt{b}} = x - \sqrt{y}.$$

For $a + \sqrt{b} = (x + \sqrt{y})^3 = x^3 + 3x^2\sqrt{y} + 3xy + y\sqrt{y}$; and, since the rational parts, as well as irrational parts, of this equation are equal,

$$\therefore a = x^3 + 3xy, \sqrt{b} = 3x^2\sqrt{y} + y\sqrt{y}; \sqrt[3]{a - \sqrt{b}} = x - \sqrt{y}.$$

The product of (1) and (2) is

$$(3) \quad \sqrt[3]{a^2 - b} = x^2 - y = c,$$

if $a^2 - b$ is a perfect cube, and $= c$.

But $a = x^3 + 3xy = x^3 + 3x(x^2 - c)$ from (3); and therefore $4x^3 - 3cx = a$, from which x may be found; and then y is known from the equation $y = x^2 - c$.

II. *Find the sum of the squares of the first n natural numbers.*

We know that $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$.

Let $S = 1^2 + 2^2 + 3^2 + \dots + n^2$; then since

$$\begin{aligned} n^3 - (n-1)^3 &= 3n^2 - 3n + 1, \\ (n-1)^3 - (n-2)^3 &= 3(n-1)^2 - 3(n-1) + 1, \\ (n-2)^3 - (n-3)^3 &= 3(n-2)^2 - 3(n-2) + 1, \end{aligned}$$

and so on to

$$\begin{aligned} 3^3 - 2^3 &= 3 \cdot 3^2 - 3 \cdot 3 + 1, \\ 2^3 - 1^3 &= 3 \cdot 2^2 - 3 \cdot 2 + 1, \\ 1^3 - 0^3 &= 3 \cdot 1^2 - 3 \cdot 1 + 1, \end{aligned}$$

we get, by adding vertically,

$$\begin{aligned} n^3 &= 3(1^2 + 2^2 + 3^2 + \dots + n^2) - 3(1 + 2 + 3 + \dots + n) + n \\ &= 3s - \frac{3n(n+1)}{2} + n. \therefore s = \frac{n(n+1)(2n+1)}{2 \cdot 2}. \end{aligned}$$

III. *Find the sum of the cubes of the first n natural numbers.* We have

$$\begin{aligned} n^4 - (n-1)^4 &= 4n^3 - 6n^2 + 4n - 1, \\ (n-1)^4 - (n-2)^4 &= 4(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1, \\ (n-2)^4 - (n-3)^4 &= 4(n-2)^3 - 6(n-2)^2 + 4(n-2) - 1, \end{aligned}$$

and so on to

$$\begin{aligned} 3^4 - 2^4 &= 4 \cdot 3^3 - 6 \cdot 3^2 + 4 \cdot 3 - 1, \\ 2^4 - 1^4 &= 4 \cdot 2^3 - 6 \cdot 2^2 + 4 \cdot 2 - 1, \\ 1^4 - 0^4 &= 4 \cdot 1^3 - 6 \cdot 1^2 + 4 \cdot 1 - 1, \end{aligned}$$

and, by adding vertically, we get

$$\begin{aligned} n^4 &= 4(1^3 + 2^3 + 3^3 + \dots + n^3) - 6(1^2 + 2^2 + 3^2 + \dots + n^2) \\ &\quad + 4(1 + 2 + 3 + \dots + n) - n. \\ &= 4s - \frac{6n(n+1)(2n+1)}{2 \cdot 3} + \frac{4n(n+1)}{2} - n. \\ \therefore 4s &= n^4 + 2n^3 + n^2, \quad \text{or} \quad s = \left(\frac{n(n+1)}{2}\right)^2. \end{aligned}$$

It follows, therefore, that

$$(1 + 2 + 3 + \dots + n)^2 = 1^3 + 2^3 + 3^3 + \dots + n^3.$$

The sums of the fourth, fifth, &c. powers of the first n natural numbers may, in succession, be found by the same process.

IV. *To find the value of a recurring decimal.* Let P denote the p figures which do not recur, Q the q figures which do recur, and s the value of the recurring decimal; then

$$s = .PQQQ\dots, \quad 10^p s = P.QQQ\dots, \quad 10^{p+q} s = PQ.QQ\dots$$

$$\text{and by subtraction } (10^{p+q} - 10^p) s = PQ - P\dots s = \frac{PQ - P}{10^{p+q} - 10^p}.$$

Now, $10^{p+q} - 10^p = 10^p (10^q - 1)$; and $10^q - 1$, when expressed by figures, in the usual way, will consist of q nines.

Hence we find the usual rule for finding the value of a recurring decimal. *Subtract the integral number consisting of the non-recurring figures*

from the integral number consisting of the non-recurring and recurring figures, and divide by a number consisting of as many nines as there are recurring figures followed by as many ciphers as there are non-recurring figures.

V. *Demonstration of the Binomial Theorem for any exponent*, supposing the development for positive integer exponents to be known.

Suppose m and n to be positive integers ; then

$$(1) \quad (1+x)^m = f(m) = 1 + mx + \frac{m(m-1)}{1.2} x^2 + \frac{m(m-1)(m-2)}{1.2.3} x^3 + \&c.$$

$$(2) \quad (1+x)^n = f(n) = 1 + nx + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 + \&c.$$

using the notation $f(m)$ and $f(n)$ for brevity. The product of (1) and (2) gives

$$(1+x)^m (1+x)^n = (1+x)^{m+n} = f(m)f(n).$$

But, since $m+n$ is a whole number, the development of $(1+x)^{m+n}$ must be of the same form as (1) or (2).

$$(3) \quad \therefore (1+x)^{m+n} = f(m+n) = f(m) \times f(n).$$

For n write p , and then change m to $m+n$; then

$$f(m+n+p) = f(m+n)f(p) = f(m) \times f(n) \times f(p).$$

Continuing this process, we shall obtain

$$(4) \quad f(m+n+p+q \dots) = f(m) \times f(n) \times f(p) \times f(q) \times \dots$$

Now, let $m = n = p = q = \dots = \frac{s}{r}$, a fraction, s and r being positive integers, and suppose the number of fractions to be r ; then (4) becomes

$$f(s) = \left(f\left(\frac{s}{r}\right)\right)^r. \quad \therefore (f(s))^{\frac{1}{r}} = f\left(\frac{s}{r}\right).$$

But since s is a positive integer, $f(s) = (1+x)^s$, and therefore

$$(f(s))^{\frac{1}{r}} = (1+x)^{\frac{s}{r}} = f\left(\frac{s}{r}\right) = 1 + \frac{s}{r} x + \frac{\frac{s}{r}(\frac{s}{r}-1)}{1.2} x^2 + \&c.$$

when the exponent is any positive fraction. Again, in (3) put $-n$ for m ; then

$$f(-n) \times f(n) = f(-n + n) = f(0) = 1,$$

as is evident from (1) or (2).

Wherefore $\frac{1}{f(n)} = f(-n)$. But for any positive value of n we have $f(n) = (1 + x)^n$.

$$\begin{aligned} \therefore \frac{1}{f(n)} &= \frac{1}{(1 + x)^n} = (1 + x)^{-n} = f(-n) \\ &= 1 + (-n)x + \frac{(-n)(-n-1)}{1 \cdot 2}x^2 + \dots, \end{aligned}$$

which shows the form of the development to be the same for negative exponents. This demonstration for fractional and negative exponents is due to EULER.

EXPOSITION OF THE PROCESS OF MATHEMATICAL DEVELOPMENT.

By JOHN PATERSON, A. M., Albany, N. Y.

[Continued from Page 294, Vol. II.]

6. DURING the time $1'$, while the generating force φ^{n-2} of the third rank has increased duplicately from 0 to the ultimate value $3\varphi^{n-2}$, its generated force of the fourth (descending) rank, φ^{n-3} , has increased in triplicate ratio; so that if $3\varphi^{n-2}$ has generated $1\varphi^{n-3}$ in the time $1'$, it will generate $3\varphi^{n-3}$ in the time $1''$. Applying this relation to the semi-units $\frac{1}{2} \cdot 1'$ and $\frac{1}{2} \cdot 1''$, instead of the units $1'$ and $1''$ of time, and recollecting that since φ^{n-3} increases in triplicate ratio during the time of its genesis, its effect in an equal time with its ultimate value constant must necessarily be m'' times its effect, while increasing triplicately from zero, it is plain that the following equations are authorized.

$$\begin{aligned}\frac{1}{2}' \cdot 1_i' + \frac{1}{2}'' \cdot 1_i' &= 1_i': \\ \frac{1}{8} \varphi_1^{n-3} + \frac{3}{8} \varphi_1^{n-3} + \frac{3}{8} \varphi_2^{n-3} + \frac{1}{8} \varphi_2^{n-3} &= 1 \varphi^{n-3}; \\ \frac{m''}{8} \varphi_1^{n-3} + \frac{3m''}{8} \varphi_1^{n-3} + \frac{3m''}{8} \varphi_2^{n-3} &= \frac{m''}{8} \varphi_2^{n-3} = m'' \varphi^{n-3};\end{aligned}$$

the first and last terms having the relation of constant to variable, as before.

The force $\frac{m''}{8} \varphi_1^{n-3}$ would generate $\frac{m''}{8} \varphi^{n-4}$ in a unit of time, and therefore does generate $\frac{m''}{16} \varphi^{n-4}$ in the time $\frac{1}{2}' \cdot 1_i'$, the m'' th part of which is $\frac{1}{16} \varphi^{n-4}$ the amount it has generated in the time $\frac{1}{2}' \cdot 1_i'$ of its own genesis, and is also the amount generated by $\frac{m''}{8} \varphi_2^{n-3}$ in the time $\frac{1}{2}' \cdot 1_i'$ of its genesis.

The force $\frac{3m''}{8} \varphi_1^{n-3}$ would generate $\frac{3m''}{8} \varphi^{n-4}$ in a unit of time, or $\frac{3m''}{16} \varphi^{n-4}$ in a semi-unit of time; but as it is of the third rank from its constant generator $\frac{3}{4} \varphi_1^{n-1}$, it increases in duplicate ratio, and therefore its successive effect must be divided by 3 to get its synchronous measure $\frac{3}{8} \cdot \frac{m''}{16} \varphi^{n-4}$ for the time $\frac{1}{2}'' \cdot 1_i'$.

The force $\frac{3m''}{8} \varphi_2^{n-3}$ would generate $\frac{3m''}{8} \varphi^{n-4}$ in a unit of time, or $\frac{3m''}{16} \varphi^{n-4}$ in a semi-unit of time; but as it is of the second rank from its constant generator $\frac{3}{4} \varphi_1^{n-2}$, it increases in simple ratio, and therefore its successive effect must be divided by 2 to get $\frac{3}{8} \cdot \frac{m''}{16} \varphi^{n-4}$, its synchronous effect in the time $\frac{1}{2}'' \cdot 1_i'$. The collected terms fetch the equation $(\frac{1}{8} + \frac{3}{8} \cdot \frac{m''}{16} + \frac{3}{8} \cdot \frac{m''}{16} + \frac{1}{8} \cdot \frac{m''}{16} + \frac{1}{8}) \varphi^{n-4} = 1 \varphi^{n-4}$, which gives $m'' = 4$,

$$\frac{1}{2}' \cdot 1_i' + \frac{1}{2}'' \cdot 1_i' = 1_i':$$

and converts the last preceding equation into

$$\frac{1}{8} \varphi_1^{n-3} + 3 \cdot \frac{1}{8} \varphi_1^{n-3} + 3 \cdot \frac{1}{8} \varphi_2^{n-3} + \frac{1}{8} \varphi_2^{n-3} = 4 \varphi^{n-3}.$$

Thus the successive and synchronous measures of a force of the fourth rank are mutually convertible by the factor 4.

NOTE. — A quantity may increase from zero by a regular law to any assigned magnitude in a given time ; and it is seen by the preceding deductions that the law of increase determines the divisor for reducing successive to synchronous measure, independently of the absolute magnitude of the former. In the last preceding case, the successive semi-unit measure of the effect of the force $3 \cdot \frac{1}{8} \varphi_2^{n-3}$ is $3 \cdot \frac{1}{16} \varphi^{n-4}$; but as this effect increases in duplicate ratio, its synchronous measure calls for the divisor 2 ; while in the same place the successive semi-unit effect of $3 \cdot \frac{1}{8} \varphi_1^{n-3}$ is also $3 \cdot \frac{1}{16} \varphi^{n-4}$, but increases in triplicate ratio, and is reduced to synchronous measure by the divisor 3.

7. The generator $4 \varphi^{n-3}$ increases in triplicate ratio, and therefore its generatant φ^{n-4} increases in quadruplicate ratio : the successive effect of this generatant, in its function as generator, will necessarily be m''' times its synchronous effect ; which authorizes the multiplication of the first of the following equations by m''' .

$$\frac{1}{2}' \cdot 1' + \frac{1}{2}'' \cdot 1' = 1' :$$

$$\frac{1}{16} \varphi_1^{n-4} + \frac{1}{16} \varphi_1^{n-4} + \frac{6}{16} \varphi^{n-4} + \frac{1}{16} \varphi_2^{n-4} + \frac{1}{16} \varphi_2^{n-4} = 1 \varphi^{n-4} ;$$

$$\frac{m'''}{16} \varphi_1^{n-4} + \frac{4m'''}{16} \varphi_1^{n-4} + \frac{6m'''}{16} \varphi^{n-4} + \frac{4m'''}{16} \varphi_2^{n-4} + \frac{m'''}{16} \varphi_2^{n-4} = m''' \varphi^{n-4}.$$

The force $\frac{m'''}{16} \varphi_1^{n-4}$ increases in quadruplicate ratio from 0, being of the fifth rank from its constant generator $1 \varphi^n$: its successive unit measure of effect becomes $\frac{m'''}{32} \varphi^{n-5}$ for the semi-unit of time $\frac{1}{2}'' \cdot 1'$, leaving $\frac{1}{32} \varphi^{n-5}$ for its synchronous measure in the time $\frac{1}{2}' \cdot 1'$, which is also the synchronous measure of the effect of $\frac{m'''}{16} \varphi_2^{n-4}$ for the time $\frac{1}{2}'' \cdot 1'$.

The force $\frac{4m'''}{16} \varphi_1^{n-4}$ increases in triplicate ratio, being of the fourth rank from its constant generator $\frac{1}{2} \varphi_1^{n-1}$: its successive semi-unit of

effect $\frac{4}{32} m''' \varphi^{n-5}$ increases quadruplicately, and must be divided by 4 to get $\frac{1}{4} \cdot \frac{m'''}{32} \varphi^{n-5}$ its synchronous effect for the time $\frac{1}{4}'' \cdot 1'$.

The force $\frac{6}{16} m''' \varphi^{n-4}$ increases in duplicate ratio, being of the third rank from the constant generator $\frac{3}{4} \varphi_1^{n-2}$: its successive semi-unit effect $\frac{6}{32} m''' \varphi^{n-5}$ increases triplicately, and must be divided by 3 to get $\frac{2}{3} \cdot \frac{m'''}{32} \varphi^{n-5}$ its synchronous effect for the time $\frac{1}{2}'' \cdot 1'$.

The force $\frac{4}{16} m''' \varphi_2^{n-4}$ increases in simple ratio, being of the second rank from the constant generator $\frac{1}{2} \varphi_1^{n-3}$: its successive semi-unit of effect $\frac{4}{32} m''' \varphi^{n-5}$ increases duplicately, and must be divided by 2 to get $\frac{1}{2} \cdot \frac{m'''}{32} \varphi^{n-5}$ its synchronous measure for the time $\frac{1}{2}'' \cdot 1'$.

The collected results constitute the equation

$$\left(\frac{1}{3} \frac{1}{2} + \frac{1}{4} \cdot \frac{m'''}{32} + \frac{2}{3} \cdot \frac{m'''}{32} + \frac{1}{2} \cdot \frac{m'''}{32} + \frac{1}{4} \cdot \frac{m'''}{32} + \frac{1}{3} \frac{1}{2} \right) \varphi^{n-5} = m''' \varphi^{n-5},$$

which gives $m''' = 5$, and converts the last preceding equation into

$$\frac{1}{3} \cdot 1' + \frac{1}{2} \cdot 1' = 1':$$

$$\frac{5}{16} \varphi_1^{n-4} + \frac{2}{16} \varphi_1^{n-4} + \frac{3}{16} \varphi^{n-4} + \frac{2}{16} \varphi_2^{n-4} + \frac{1}{16} \varphi_2^{n-4} = 5 \varphi^{n-4}.$$

Thus the successive and synchronous measures of a force of the fifth rank are mutually convertible by the factor 5.

8. So much may suffice to elucidate the genetical process occurring during the first unit of time; which process may be prolonged downwards to the n th rank, terminating with $\varphi^{n-n} = \varphi^0$, the force of the order zero which is no generator.

9. For $n = 4$, the two series of synchronous and successive unit measures of the generated forces, including the primitive generator, are

$$1 \varphi^n, 1 \varphi''', 1 \varphi'', 1 \varphi', 1 \varphi^0,$$

and

$$1 \varphi^n, 2 \varphi''', 3 \varphi'', 4 \varphi', 1 \varphi^0;$$

each term of the first series being the measure of the generated effect confined throughout to the first unit of time, while the coefficient of each term of the second series expresses the relation between the effects of its efficient in the times $1'_i$ and 1_i . Thus, while the synchronous effect of the force $3\varphi''$ for the time 1_i is $1\varphi'$ referred to synchronous effect of the latter force, it is $4\varphi'$ in successive measure, and the successive effect of the same force $3\varphi''$ is $3.4\varphi'$ for the time $1'_i$.

The genetical operations for the first two units of time are as follows : —

$1'_i$.	$1''_i$.
$1\varphi'' = 1\varphi'' :$	$\frac{1.2}{1}\varphi''', \frac{1.2.3}{1.2}\varphi'', \frac{1.2.3.4}{1.2.3}\varphi', \frac{1.2.3.4.1}{1.2.3.4}\varphi^0 = 1\varphi^0.$
$1\varphi''' = 2\varphi''' :$	$\frac{2.3}{1}\varphi'', \frac{2.3.4}{1.2}\varphi', \frac{2.3.4.1}{1.2.3}\varphi^0 = 4\varphi^0.$
$1\varphi'' = 3\varphi'' :$	$\frac{3.4}{1}\varphi', \frac{3.4.1}{1.2}\varphi^0 = 6\varphi^0.$
$1\varphi' = 4\varphi' :$	$\frac{4.1}{1}\varphi^0 = 4\varphi^0.$
$1\varphi^0 = 1\varphi^0.$	

The final effect of the force $1\varphi''$ in two units of time is

$$(1 + 4 + 6 + 4 + 1)\varphi^0 = 16\varphi^0 = 2^4\varphi^0.$$

In the first left-hand column, each term below the first expresses the synchronous value of the effect generated by the immediately superior term of the second column (directly on the right of the equality signs), during the time $1'_i$; each term of the second column increasing from 0 to the value expressed by its coefficient, and being at the same time the successive value of the effect generated by its immediately superior term of the same column.

Passing to the right of the colon points, the primitive force $1\varphi''$ generates in the second unit of time $1''_i$, a series equal to that of $1'_i$, closing with $\frac{1.2.3.4.1}{1.2.3.4}\varphi^0 = 1\varphi^0.$

The force $2\varphi'''$, now become a constant generator, having generated $3\varphi''$ while increasing uniformly during $1'_t$, will generate $2.3\varphi''$ in the time $1''_t$: this force $2.3\varphi''$ increases uniformly during $1''_t$, and therefore its successive measure $2.3.4\varphi'$ must be divided by 2 to get its synchronous value $\frac{2.3.4}{1.2}\varphi'$; and since the force $\frac{2.3.4}{1.2}\varphi'$ increases duplicately, and is of the first order, its successive effect, $\frac{2.3.4.1}{1.2}\varphi^0$, has no successive value, but must nevertheless be divided by 3 to get its synchronous value, $\frac{2.3.4.1}{1.2.3}\varphi^0 = 4\varphi^0$.

The force $3\varphi''$, now become a constant generator, and having generated $4\varphi'$ while increasing duplicately during $1'_t$, will generate $3.4\varphi'$ in the time $1''_t$: this force $3.4\varphi'$ increases uniformly during $1''_t$; and as it is of the first order, its successive effect, $3.4.1\varphi^0$, has no successive value, but must be divided by 2 to fetch its synchronous value $\frac{3.4.1}{1.2}\varphi^0 = 6\varphi^0$.

Finally, the force $4\varphi'$, now become a constant generator, and being of the first order, its successive effect, 4.1φ , contains no element of increase, or is uniform and without successive value, and therefore has but the divisor 1 to bring its synchronous value $\frac{4.1}{1}\varphi^0 = 4\varphi^0$ for the time $1''_t$.

10. If, then, a unit force $1\varphi^{iv}$ of the fourth order be applied to a unit of mass 1_μ during a unit of time 1_t , at the expiration of the first unit interval $1'_t$ the mass 1_μ has described a unit of distance $1_t = 1\varphi^0$ the unit force of the order zero, the phenomenal measure of the primitive force in and for the first unit of time: there is in the same time $1'_t$ also generated the velocity 4.1_λ or force $4\varphi'$ of the first order, which will, by uniform motion, carry the unit mass 1_μ the distance 4.1_t in the next unit of time $1''_t$; also the forces $2\varphi'''$ and $3\varphi''$, which each generate the velocity 12.1_λ in the time $1''_t$, but under different rates of acceleration, such as suffice to carry a unit mass respectively the distances 4.1_t and 6.1_t in the time $1''_t$ of their genesis; to these amounts add the unit of distance again gen-

erated by the primitive force $1\varphi^n$ in the second unit of time, and there is generated the distance 1_1 in the first unit of time; and in the second unit of time, the distance 4.1_1 by the velocity in existence at the beginning of this interval, and 11.1_1 by the forces in operation; making 15.1_1 for the time $1''$, and in all $16.1_1 = (1+1)^4 1_1$ for the whole time 2.1_1 .

11. As the interval of time is arbitrary, it may be taken equal to $\psi 1$, or $h 1$, and so achieve the transition from $(1+1)^n$ to ψ^n and $(\psi+h)^n$, etc.

As the force $1\varphi^n$ generates $2\varphi'''$ uniformly in the time 1_1 , it will generate $2\psi\varphi'''$ uniformly in the time $\psi 1_1$.

As the force $2\varphi'''$ generates $2.3\varphi''$ uniformly in the time $1'_1$ and $3\varphi''$ duplicately in the preceding equal time $1'_1$, so the force $2\psi\varphi'''$ will generate $2.3\psi^2\varphi''$ uniformly in the time $\psi'' 1_1$ and $3\psi^2\varphi''$ duplicately in the preceding equal time $\psi' 1_1$.

As the force $3\varphi''$ generates $3.4\varphi'$ uniformly in the time $1'_1$ and $4\varphi'$ triplicately in the time $1'_1$, so the force $3\psi^2\varphi''$ will generate $3.4\psi^3\varphi'$ uniformly in the time $\psi'' 1_1$ and $4\psi^3\varphi'$ triplicately in the preceding equal time $\psi' 1_1$.

As the force $4\varphi'$ generates $4\varphi^0$ uniformly in the time $1'_1$ and $1\varphi^0$ quadruplicately in the time $1'_1$, so the force $4\psi^3\varphi'$ will generate $4\psi^4\varphi^0$ uniformly in the time $\psi'' 1_1$ and $1\psi^4\varphi^0$ quadruplicately in the preceding equal time $\psi' 1_1$.

[To be Continued.]

Mathematical Monthly Notices.

Cambridge Senate-House Problems and Riders for the Year 1860; WITH SOLUTIONS. By the REV. H. W. WATSON, M. A., late Fellow of Trinity College; and E. J. ROUTH, M. A., Fellow and Assistant Tutor of St. Peter's College, Cambridge, and Examiner in the University of London. Cambridge: Macmillan & Co., and 23 Henrietta Street, Covent Garden, London. 1860. pp. 198. 3 plates containing 55 cuts.

The Moderators for the year 1860 are the Editors of the volume; the Examiners are PERCIVAL FROST, M. A., St. John's College, and NORMAN MACLEOD FERRERS, M. A., Gonville

and Caius College. The examples alone, collected and arranged in this series of volumes, form a valuable annual contribution to mathematical science; but their value, especially to students, is greatly increased, when, as in this volume, the problems are accompanied with solutions by those who proposed them. We earnestly invite the attention of teachers and students to this volume, not only for what it contains, but also as indicating the standard in the great mathematical University.

A Treatise on Plane Co-ordinate Geometry, as applied to the Straight Line and the Conic Sections, With numerous Examples. By I. TODHUNTER, M. A., Fellow and Assistant Tutor of St. John's College, Cambridge. Second edition revised. Cambridge: Macmillan & Co. 1858. pp. 316.

This is one of a series of eight or nine volumes, prepared by the author for the use of schools and colleges. Great care has been taken in the methods of demonstration and in the explanations to adapt them to the wants of beginners. There is no branch of mathematics in which we discover such important changes and improvements in the text-books, in use in Great Britain, as in Analytic Geometry. We have already called the attention of our readers to two valuable works upon this subject (SALMON'S and PUCKLE'S); and now we are able to say, that, after a careful examination, we can heartily commend the one before us. Besides the usual propositions upon the straight line and conics, it contains an exposition of many of the new processes more recently introduced into elementary works, leaving no doubt of their having secured a permanent place in elementary instruction. Each chapter is followed by a large number of appropriate examples, and at the end of the volume we find all the answers collected together, with hints for the solution of many of those which appear more difficult.

Algebra for the Use of Colleges and Schools. With numerous Examples. By I. TODHUNTER, M. A., Fellow of St. John's College, Cambridge. Second edition revised. Macmillan & Co., Cambridge, and 23 Henrietta Street, Covent Garden, London. 1860. pp. 516.

We have examined this work with unusual interest, because it is the first recent work upon the subject, useful abroad, which has fallen into our hands, thus enabling us to compare it with works at home filling the same place in our system of instruction. We notice, first, that the general theory of equations is entirely omitted, and also the solution of equations of all degrees above the second; nor does the work include any discussion making it necessary to anticipate, by definition or otherwise, even the elements of the higher parts of the course; and second, the introduction of many subjects not usually treated in the Elements of Algebra. An enumeration of some of them will show the extent and direction of this innovation; a few subjects usually treated being included on account of the clearness and fulness with which they are developed. Theory of *Quadratic Expressions* in addition to Quadratic Equations; Imaginary Expressions; Ratio introductory to Proportion; Variation; Scales of Notation; Harmonical Progression; Mathematical Induction; Multinomial Theorem; Logarithms, with Exponential and Logarithmic Series; Convergency and Divergency of Series; Indeterminate Equations of the First and Second Degrees; Partial Fractions and Indeterminate Coefficients; Recurring Series; Summation of Series; Inequalities; Theory of Numbers; Probability; Miscellaneous Theorems. All these subjects are treated in a very plain and familiar style; and the whole volume possesses many of the best traits of a good text-book. The author says: "Students should be encouraged to examine carefully the language of the book they are using, so that they may ascertain its meaning, or be able to point out exactly where their difficulties arise. The language, therefore, ought to be simple and precise; and it is essential that apparent conciseness should not be gained at the expense of clearness." Students, who are pursuing their mathematical studies without the aid of a teacher, will find this work well adapted to their wants. It contains about sixteen hundred and fifty examples, the best collection for the school-room we have ever seen.

University Algebra; embracing a Logical Development of the Science, with numerous graded Examples. By CHARLES DAVIES, LL. D., Professor of Mathematics in Columbia College. New York: A. S. Barnes and Burr, 51 and 53 John Street. 1860. pp. 320.

Our acquaintance with all of the author's more advanced books, both as a student and teacher, and our conviction that they have accomplished a vast good to the cause of sound education in this country, makes us feel a deep interest in every new addition to the series. A careful examination of the work before us shows that it is intended to occupy a place intermediate between the first elements of algebra and advanced treatises, a fact which the term *University* might not indicate to all; and, that it is one of the most carefully prepared works of the author's series, possessing so many of the requisites of a good text-book, as to insure it an extensive introduction as soon as its merits shall become known. Besides the fact that the definitions, demonstrations, and explanations are all expressed with great clearness, and in that happy mean between extreme conciseness and prolixity, which constitutes a model didactic style in mathematical writing, we beg to note the following special points. The demonstration of the Binomial Theorem for positive integral powers is well adapted for beginners. A few of the successive powers of a binomial are found by actual multiplication from which the general laws of exponents and coefficients are inferred; these laws are then assumed for the m th power, and, shown to hold for the $(m+1)$ st power, a species of proof known as mathematical induction.

The case of the Problem of the Lights discussed in the Monthly, Vol. I. pp. 160, 178, and also in the present number, is thus disposed of: "The hypothesis of $c = 0$, places the two lights at the same point, in which case they form one and the same light, whose intensity is equal to the sum of their intensities taken separately. The conditions of the problem involve the necessity of *two lights*, and the equation of the problem is found under this hypothesis. This equation ought not, therefore, to respond to the case of a single light. For the interpretation of the results obtained from an equation can only give the cases which fall under the hypothesis. The hypothesis of two lights, and the hypothesis of a single light, are not connected by any law which affords a common equation of condition. Hence, the results obtained on the supposition $c = 0$ do not belong to the problem."

It certainly is possible, mathematically, to suppose that two lights of unequal intensities may occupy the same point and still be regarded as separate and distinct lights; and the question may properly be asked whether two such lights, so situated and regarded, can illuminate any point in space by the same amount. It seems to us, therefore, that it would have been better to have shown, that, under this hypothesis, the condition that a point should be equally illuminated by the two lights becomes impossible. This would have obviated a rather doubtful explanation of the difficulty.

The chapter on the General Theory of Equations is brief, but as full as is consistent with the plan of the work. We notice that derived equations are given by simple definition, no attempt at an exposition of the principles involved being made. This seems to suggest whether it is not better to postpone the general theory of equations till later in the course, after the student shall have acquired, at least the elements of the calculus. The great fault with many of our text-books is, that they encroach upon each other's ground; and thus consume too much of the student's time by simple repetition.

After giving the form of the general equation of the n th degree, containing but one unknown quantity, with positive or negative, real or imaginary coefficients, the author says: "It is evident that this equation has at least one root, for if the first member is equal to 0, it must be so for some value of x , either real or imaginary, and this value of x is, by definition, a root." The only difficulty in this disposition of a proposition, which has lately received much attention from eminent mathematicians, is, that it assumes the point to be proved. *If a function of x equals*

zero, as a matter of course it must be so for some value of x , either real or imaginary. But is there necessarily some value of x of the form of $a + b\sqrt{-1}$ which will make the function zero? This is the question.

Astronomical Notices. — No. 21. Edited by DR. F. BRÜNNOW, Professor of Astronomy and Director of the Observatory of Michigan University, Ann Arbor. Ann Arbor, October 11th, 1860.

The contents of this number are, a letter from R. F. BOND, Esq., on a New Clock-Escape-ment, with a cut; a letter from COMMANDER MAURY, announcing the discovery of a new Asteroid, September 14th, by MR. FERGUSON, with Observations of Leucothea, Comet III., 1860, and Solar Eclipse, July 18th; Co-ordinates of Spots visible on the Sun, July 18th, and Observation of the Solar Eclipse by DR. C. H. F. PETERS, Director of the Observatory of Hamilton College.

A Collection of Problems and Examples in Mathematics; selected from the Jesus College Examination Papers, with Answers. — By H. A. MORGAN, M. A., Sadlerian and Mathematical Lecturer of Jesus College, Cambridge. Cambridge: Macmillan & Co. 1858. pp. 190.

This is a collection of simple examples for the use of students, arranged as follows, with the number under each head: Euclid, 55; Conic Sections, Parabola, 27; Ellipse, 29; Hyperbola, 6; Algebra, 66; Theory of Equations, 23; Trigonometry, 96; Analytical Geometry of Two Dimensions, 94; Differential and Integral Calculus, 70; Differential Equations, 15; Geometry of Three Dimensions, 26; Elementary Statics, 61; Analytical Statics, 47; Elementary Dynamics, 57; Newton, Books I., II., III., 54; Dynamics of a Particle, 33; Rigid Dynamics, 32; Hydrostatics, 57; Optics, 72; Spherical Trigonometry, and Astronomy, 32; Answers to the Questions.

Editorial Items.

PRIZE SOLUTIONS. — Owing to the delay in the publication of the September No., Vol. II., and October No., Vol. III., one month additional will be given for the solutions of the problems in these numbers. We trust that students will continue their interest in this feature of the Monthly, and that teachers will encourage their pupils to send us solutions.

LAPLACE'S *Autograph*. — It will be seen that there is no doubt about the genuineness of this Autograph. Very little of LAPLACE'S manuscript is still in existence, and we thank MR. WATSON for enabling us to present copies of this interesting note to our readers.

MATHEMATICAL TEXT-BOOKS. — We propose to give more attention than formerly to Elementary Text-books in Mathematics, Natural Philosophy, &c. A proper proportion of each number will be thus devoted.

ERRATA. — On page 419, Vol. II., line 10, for x^2 read $n x^2$; p. 421, line 7, read $R = \frac{p}{n} \sqrt{1-n}$, and on line 14, read $R = \sqrt{A^2 - B^2}$.

MR. EVANS has called our attention to the misstatement of Problem V. in the September Number. It should read, "Prove that a chord joining one pair of these points will meet two other such chords each at a point on an asymptote, and be in the case of the circle perpendicular to one of them."

THE
MATHEMATICAL MONTHLY.

Vol. III . . . DECEMBER, 1860. . . . No. III.

PRIZE PROBLEMS FOR STUDENTS.

I. From AC , the diagonal of a square $ABCD$, cut off AE equal to one fourth of AC , and join BE , DE . Show that the figure $BADE$ equals twice the square on AE .

II. Two toothed wheels work against each other. Show that, if the number of teeth in one be prime to that in the other, before two teeth, which have once been in contact, come in contact again, every tooth of the one wheel will have been in contact with every tooth of the other.

III. Eliminate φ from the equations

$$\begin{aligned}x \cos (\varphi + \alpha) + y \sin (\varphi + \alpha) &= a \sin 2 \varphi, \\y \cos (\varphi + \alpha) + x \sin (\varphi + \alpha) &= 2 a \cos 2 \varphi,\end{aligned}$$

showing that

$$(x \sin \alpha - y \cos \alpha)^2 + (y \sin \alpha + x \cos \alpha)^2 = (2 a)^2.$$

IV. Show that $\frac{1}{4} m^2$ is the area of the greatest triangle which can be formed with the lines a , b , c , subject to the condition $a^2 + b^2 + c^2 = 3 m^2$.

V. The equation of a family of ellipses is $a x^2 + b y^2 = 1$, when $a - b = c$, a constant. Show that the curve, which cuts all the individuals at an angle whose tangent is $\frac{y}{x}$, is $x = C e^{-\frac{c y^2}{2}}$.

Solutions of these problems must be received by February 1, 1861.

NOTES AND QUERIES.

1. *Perfect Squares*.—It is a well-known fact that every perfect square, when represented by the Arabic or Hindoo method, must end in 0, 1, 4, 5, 6, or 9. This fact enables us to decide, at a glance, that all numbers ending in 2, 3, 7, or 8 are not perfect squares; but it does not aid us in deciding which of the numbers ending in 0, 1, 4, 5, 6, or 9 are perfect squares, for the converse of the proposition stated above is not true. To assist, to some extent, in removing this difficulty, I submit the following discussion.

PROPOSITION.—The last two figures of the square of any integer must correspond to the last two figures of the square of some number in the series 1, 2, . . . 99. The endings of the squares of the first twenty-five numbers form the series 01, 04, 09, 16, 25, 36, 49, 64, 81, 00, 21, 44, 69, 96, 25, 51, 89, 24, 61, 00, 41, 84, 29, 71, 24. This series also represents the endings of the squares of the numbers from 51 to 75, inclusive. The series, taken backwards, represents the endings of the numbers from 25 to 49, and from 75 to 99, inclusive. This shows that the squares of the numbers equally above and below 25 and 75 end in the same two figures. Omitting the like endings in the above series, and arranging in order, we have 00, 01, 04, 09, 16, 21, 24, 25, 29, 36, 41, 44, 49, 56, 61, 64, 69, 76, 81, 84, 89, 96. These endings indicate the following facts: Every perfect square ending in 1, 4, or 9 must have before the last figure one of the even numbers 0, 2, 4, 6, or 8; ending in 6, must be preceded by one of the odd numbers 1, 3, 5, 7, or 9; ending in 5, by 2; and ending in 0, by 0.—Prof. W. D. HENKLE, Normal School, Lebanon, Ohio.

2. *Problem*.—Prove that, when x is a whole number and greater than one, $x^3 - x$ is divisible by 6.

Suppose that $x^3 - x$ is divisible by 6 when $x = p$, then will it be divisible by 6 when $x = p + 1$. For, letting $x = p + 1$, we have

$$x^3 - x = p^3 + 3p^2 + 3p + 1 - (p + 1) = p^3 + 3p^2 + 2p = p^3 - p + 3p^2 + 3p.$$

Now $p^3 - p$ is divisible by 6, by hypothesis; and if we can prove that $3p^2 + 3p$ is also divisible by 6, then it will follow that $x^3 - x$ is divisible by 6 when $x = p + 1$. But to prove that $3p^2 + 3p$ is divisible by 6, we must prove that $p^2 + p$ is divisible by 2. Suppose that $p^2 + p$ is divisible by 2 when $p = r$, then will it be divisible by 2 when $p = r + 1$. For, putting $p = r + 1$, we get

$$p^2 + p = r^2 + 2r + 1 + r + 1 = r^2 + 3r + 2 = r^2 + r + 2r + 2.$$

But $r^2 + r$ is divisible by 2 by hypothesis, and $2r + 2$ is also divisible by 2. Now we know that $p^2 + p$ is divisible by 2 when $p = 1$, $\therefore$ it must be when $p = 2, 3, 4$, &c.; and therefore if $x^3 - x$ is divisible by 6 when $x = p$, it will be when $x = p + 1$. But we know by inspection that it is divisible when $x = 2$; $\therefore$ it will be when $x = 3, 4, 5$, &c. — JOHN SCOTT, Jr., Night High School, Cincinnati, Ohio.

3. *Note on the Prize Solution of Problem III.*, p. 25. — As there has been a slight typographical error each time the formulas have been printed, I send them in their correct and most reduced form. As m and n exchange values in the two solids, the formulas take very nearly the same forms, one having cosine where the other has the square of the cotangent, and 3 in the place of 2. The last forms given below are adapted to logarithmic computation.

$$\text{Vol. Dodec.} = \frac{5 e^3 \cot^2 \beta}{2 \sqrt{(4 \sin^2 \beta - 1)}} = e^3 \left(\frac{15 + 7 \sqrt{5}}{4} \right) = \frac{5 e^3 \tan^2 54^\circ}{4 \sqrt{(\sin 6^\circ \cos 24^\circ)}}.$$

$$\begin{aligned} \text{Vol. Icos.} &= \frac{5 e^3 \cos \beta}{3 \sqrt{(3 - 4 \cos^2 \beta)}} = \frac{5 e^3 \cos \beta}{3 \sqrt{(4 \sin^2 \beta - 1)}} \\ &= e^3 \left(\frac{15 + 5 \sqrt{5}}{12} \right) = \frac{5 e^3 \sin 54^\circ}{6 \sqrt{(\sin 6^\circ \cos 24^\circ)}}. \end{aligned}$$

— PROF. D. W. HOYT, New Hampton Institution, Fairfax, Vt.

4. *Correction of the Method of Computing Six per Cent Interest for Days.*

— The usual method of computing interest at six per cent, for a given

number of days, that is, “ Divide the principal by 6,000 and multiply the quotient by the given number of days,” is not of course accurate when large numbers are involved ; but the following development, p , denoting the principal, and d the days.

$$\frac{0.06 p d}{365.25} = \frac{p d}{6000} \left\{ 1 + \frac{1}{10} \left[1 + \frac{1}{10} \left(1 + \frac{1}{10} \right) \right] \right\},$$

shows that the correction is almost exactly $\frac{1}{10}$ of the first result; and if this is not sufficient, the second correction is $\frac{1}{10}$ of the first. The third correction, $\frac{1}{10}$ of the second, will never be needed, and is not quite so convenient to apply mentally as the others, but it is accurate to 0.000001 of its value. It will be noticed that these corrections are all additive. All except the first, however, are more curious than useful — B.

5. *Solution of Problem V., No. IX., Vol. II.* — Two great circles are drawn at random on a sphere. What is the probability that their mutual inclination, taken less than 90° , will be contained between any given limits, as n° and m° ?

1st. The mutual inclination of the two circles will be equal to the angular distance of their poles.

2d. The pole of either great circle is just as likely to fall in one position as in any other; hence the probability that it will fall in any given portion of the surface of the sphere is proportional to the superficial area of that portion.

3d. In order that the angular distance of the poles may be between the limits n° and m° , one pole must fall within the zone contained between two small circles at the respective distances of n° and m° from the other pole.

4th. Hence the probability required will be equal to the area of this zone divided by the area of the hemisphere, or to $\cos n^\circ - \cos m^\circ$. LAPLACE, in considering the inclinations of the orbits of the planets

and comets, in the *Théorie Analytique des Probabilités*, fell into the error of supposing that this probability would be equal to $\frac{m^\circ - n^\circ}{90^\circ}$; or that all inclinations would be equally probable. — S. NEWCOMB, Cambridge, Mass.

6. *Note on the Powers of Binomials.* — Find the fifth power of the binomial $\frac{1}{2}x - 3y^2$.

1	+ 5	+ 10	+ 10	+ 5	+ 1,	Coefficients.
$\frac{1}{32}x^5$	$+\frac{1}{16}x^4$	$+\frac{1}{8}x^3$	$+\frac{1}{4}x^2$	$+\frac{1}{2}x$	+ 1,	Power of $\frac{1}{2}x$.
1	— $3y^2$	+ $9y^4$	— $27y^6$	+ $81y^8$	— $243y^{10}$,	Powers of $-3y^2$.
$\frac{1}{32}x^5 - \frac{1}{16}x^4y^2 + \frac{1}{8}x^3y^4 - \frac{1}{4}x^2y^6 + \frac{1}{2}xy^8 - 243y^{10} = (\frac{1}{2}x - 3y^2)^5.$						

This simple method of writing powers of complicated binomials is sufficiently obvious. Writing the different lines, and then multiplying together the terms which stand in the same column, so divides up the process, that the whole is made plain and easy to the student. — DAVIES'S University Algebra.

7. *The arithmetic mean of any number of positive quantities is greater than the geometric mean ;* that is,

$$\frac{a + b + c + d \dots + k}{n} > (a b c d \dots k)^{\frac{1}{n}}.$$

Let P denote $(a b c d \dots k)^{\frac{1}{n}}$, and Q denote $\frac{a + b + c + d \dots + k}{n}$. Suppose a and b respectively the greatest and least of the n quantities $a, b, c, d, \dots k$; and let $a_1 = b_1 = \frac{1}{2}(a + b)$, and let $P_1 = (a_1 b_1 c d \dots k)^{\frac{1}{n}}$; then, since $a_1 b_1 > a b$, we have $P_1 > P$. Next, if the factors in P_1 be not all equal, remove the greatest and least of them, and put in their places two new factors, each equal to half the sum of those removed; let P_2 denote the new geometrical mean; then $P_2 > P_1$. If we proceed in this way, we obtain a series, $P, P_1, P_2, P_3, \dots P_r$, each term of which is greater than the preceding term; and by

taking r large enough, we may have the factors of P , as nearly equal as we please ; thus, when r is large enough, we may consider $P = Q$, therefore P is less than Q . — TODHUNTER'S Algebra.

8. *Note on the Successive Derivatives of $\tan \varphi$.* — We have

$$D \tan \varphi = D \frac{\sin \varphi}{\cos \varphi} = \frac{\cos^2 \varphi + \sin^2 \varphi}{\cos^2 \varphi} = \frac{1}{\cos^2 \varphi}.$$

$$D^2 \tan \varphi = -\frac{2 D \cos \varphi}{\cos^3 \varphi} = \frac{2 \sin \varphi}{\cos^3 \varphi} = \frac{2 \tan \varphi}{\cos^2 \varphi} = 2 \tan \varphi D \tan \varphi.$$

For brevity, let us adopt the notation $\tan \varphi = D^0$, $D \tan \varphi = D^1$, $D^2 \tan \varphi = D^2$, &c. Then

$$D^2 \tan \varphi = 2 D^1 D^0,$$

$$D^3 \tan \varphi = 2 D^2 D^0 + 2 (D^1)^2,$$

$$D^4 \tan \varphi = 2 D^3 D^0 + 2 D^2 D^1 + 4 D^2 D^1 = 2 D^3 D^0 + 6 D^2 D^1,$$

$$\begin{aligned} D^5 \tan \varphi &= 2 D^4 D^0 + 2 D^3 D^1 + 6 D^3 D^1 + 6 (D^2)^2 \\ &= 2 D^4 D^0 + 8 D^3 D^1 + 6 (D^2)^2, \end{aligned}$$

$$\begin{aligned} D^6 \tan \varphi &= 2 D^5 D^0 + 2 D^4 D^1 + 8 D^4 D^1 + 8 D^3 D^2 + 12 D^3 D^2 \\ &= 2 D^5 D^0 + 10 D^4 D^1 + 20 D^3 D^2. \end{aligned}$$

In general,

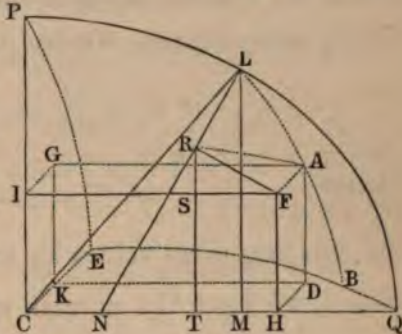
$$\begin{aligned} D^n \tan \varphi &= (D^1 + D^0)^{n-1} \\ &= D^{n-1} D^0 + (n-1) D^{n-2} D^1 + \frac{(n-1)(n-2)}{1 \cdot 2} D^{n-3} D^2 + \frac{(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3} D^{n-4} D^3 + \&c. \end{aligned}$$

In this series, terms equally distant from the extremes are equal, and may be added together. These very curious and interesting forms are found in PEIRCE'S *Curves and Functions*, Vol. I.

THE EARTH CONSIDERED AS A SPHEROID OF REVOLUTION.—GEODETIC FORMULAS.

By DR. R. C. MATTHEWSON, U. S. Deputy Surveyor, San Francisco.

LET $CEPQ$ represent one eighth of the spheroid, bounded by the convex surface EPQ , the plane of the equator EQC , and any other two planes, CPE and CPQ , at right angles to each other, passing through the pole P , and the centre C . On the plane CPQ ,* from any point L in the meridian PQ , draw the radius LC , the normal LN , and the ordinate LM . Let the meridional plane CPQ be intersected in L by the normal plane $LN B$, making with it any angle BLQ , BL being the intersection with the surface. Then the curve BL , as well as PQ , will be the arc of an ellipse. In BL take any point A , and from it let fall the perpendiculars AD , AF , and AG respectively upon the planes CQE , CPE , and CPQ , and complete the parallelograms AH , AI , HK , and IK . On the plane CPQ draw FR at right angles to LN , and RT at right angles to CQ , intersecting FI in S , and join AR .



Put $CM = x$ and $LM = y$, the rectangular co-ordinates of the point L of the meridional ellipse PQ , C being the origin; $NR = x'$ and $AR = y'$, the rectangular co-ordinates of the point A of the normal ellipse BL , N being the origin; $AD = x''$, $AF = y''$, and $AG = z''$, the rectangular co-ordinates of the point A of the spheroidal surface EPQ , C being the origin; $CQ = a$, the equatorial

* For sake of distinction, the lines upon the plane CPQ are continuous, while those not upon it are dotted.

radius; $CP = b = a(1 - e^2)^{\frac{1}{2}}$, the semi-axis, e being the eccentricity; $LN = n$, the normal; $MN = s$, the subnormal; $CN = m$, the adnormal; $CL = r$, the radius of the spheroid at the point L ; the angle $LN M = l$, the geographical latitude of the point L ; the angle $LC M = l'$, the geocentric latitude of the same point and the angle $AL Q = z$, the azimuth of the point A from L .

Then the equation of the meridional ellipse PQ is

$$a^2 y^2 + b^2 x^2 = a^2 b^2,$$

which, by eliminating b , becomes $y^2 = (1 - e^2)(a^2 - x^2)$. Differentiating this equation, we find the length of the subnormal,

$$s = y \frac{dy}{dx} = (1 - e^2) x = n \cos l.$$

Hence $x = \frac{n \cos l}{1 - e^2}$, and $y = n \sin l$.

Substituting these values for x and y in the equation of the meridional ellipse, we have the normal, $n = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 l)^{\frac{1}{2}}}$; and consequently the abscissa, $x = \frac{a \cos l}{(1 - e^2 \sin^2 l)^{\frac{1}{2}}}$; the ordinate, $y = \frac{a(1 - e^2) \sin l}{(1 - e^2 \sin^2 l)^{\frac{1}{2}}}$;

the subnormal, $s = (1 - e^2) x = \frac{a(1 - e^2) \cos l}{(1 - e^2 \sin^2 l)^{\frac{1}{2}}}$;

the adnormal, $m = x - (1 - e^2) x = e^2 x = \frac{a e^2 \cos l}{(1 - e^2 \sin^2 l)^{\frac{1}{2}}}$;

the radius, $r = (x^2 + y^2)^{\frac{1}{2}} = a \left(1 - \frac{e^2(1 - e^2) \sin^2 l}{1 - e^2 \sin^2 l}\right)^{\frac{1}{2}}$;

and the geocentric latitude,

$$l' = \frac{s}{x} \tan l = \frac{(1 - e^2) x}{x} \tan l = (1 - e^2) \tan l.$$

Again, the equation of the spheroidal surface EPQ is

$$a^2 x'^2 + b^2 y'^2 + b^2 z'^2 = a^2 b^2,$$

or, by eliminating b , $x'^2 + (1 - e^2)(y'^2 + z'^2 - a^2) = 0$. Now

in the right-angled triangle AFR we have the hypotenuse $AR = y'$, and the angle $ARF = z$, whence $AF = y' \sin z$ and $FR = y' \cos z$; in the right-angled triangle FRS we have the hypotenuse $FR = y' \cos z$ and the angle $FRS = l$, whence $FS = y' \sin l \cos z$ and $RS = y' \cos l \cos z$; and in the right-angled triangle $NR'T$ we have the hypotenuse $NR = x'$ and the angle $RNT = l$, whence $RT = x' \sin l$ and $NT = x' \cos l$. Hence,

$$x'' = AD = IC = RT - RS = x' \sin l - y' \cos l \cos z;$$

$$y'' = AF = CK = y' \sin z;$$

$$z'' = AG = CH = FS + NT + CN = y' \sin l \cos z + x' \cos l + m.$$

Now if we substitute the foregoing values for x'' , y'' , and z'' in the equation of the spheroidal surface EPQ , we shall find the equation of the normal ellipse BL to be

$$Ax'^2 + By'^2 + Cx'y' + Dx' + Ey' + F = 0,$$

$$\text{in which } A = 1 - e^2 \cos^2 l; \quad B = 1 - e^2 + e^2 \cos^2 l \cos^2 z;$$

$$C = -2e^2 \sin l \cos l \cos z; \quad D = \frac{2ae^2(1-e^2)\cos^3 l}{(1-e^2\sin^2 l)^{\frac{1}{2}}};$$

$$E = \frac{2ae^2(1-e^2)\sin l \cos l \cos z}{(1-e^2\sin^2 l)^{\frac{1}{2}}}; \quad \text{and} \quad F = -(1-e^2)(a^2 - m^2).$$

Differentiating the equation of the normal ellipse BL twice, and, in each of the differential coefficients, making $x' = n = \frac{a(1-e^2)}{(1-e^2\sin^2 l)^{\frac{1}{2}}}$ and $y' = 0$, the values of the co-ordinates when the point A coincides with L , we have

$$\frac{dx'}{dy'} = 0, \quad \text{and} \quad \frac{d^2x'}{dy'^2} = -\frac{2D}{2Ax' + D}.$$

Substituting these values for $\frac{dx'}{dy'}$, and $\frac{d^2x'}{dy'^2}$ in the general expression for the radius of the osculating circle, and then substituting the above values of A , B , D , and x' in the result, we have

$$\begin{aligned}
 R_z &= -\frac{\left(1 + \frac{d^2 x^2}{d y^2}\right)^{\frac{1}{2}}}{\frac{d^2 x}{d y^2}} = \frac{A x' + \frac{1}{2} D}{B} = \frac{a(1-e^2)}{(1-e^2 + e^2 \cos^2 l \cos^2 z)(1-e^2 \sin^2 l)^{\frac{1}{2}}} \\
 &= \frac{a(1-e^2)}{(1-e^2 \sin^2 z - e^2 \sin^2 l \cos^2 z)(1-e^2 \sin^2 l)^{\frac{1}{2}}} \\
 &= \frac{a(1-e^2)}{(\sin^2 z + \cos^2 z - e^2 \sin^2 z - e^2 \sin^2 l \cos^2 z)(1-e^2 \sin^2 l)^{\frac{1}{2}}} \\
 &= \frac{a(1-e^2)}{(1-e^2)(1-e^2 \sin^2 l)^{\frac{1}{2}} \sin^2 z + (1-e^2 \sin^2 l)^{\frac{1}{2}} \cos^2 z},
 \end{aligned}$$

which is the general expression for the radius of curvature of any normal ellipse on the spheroid making, at any latitude l , any angle z with the meridian.

When the normal ellipse BL coincides with the meridian, the angle $z = 0$, $\sin z = 0$, $\cos z = 1$, and R_z becomes $R_m = \frac{a(1-e^2)}{(1-e^2 \sin^2 l)^{\frac{1}{2}}}$, which is the radius of curvature of the meridian at the latitude l . When the normal ellipse BL is at right angles to the meridian, the angle $z = 90^\circ$, $\sin z = 1$, $\cos z = 0$, and R_z becomes $R_n = \frac{a}{(1-e^2 \sin^2 l)^{\frac{1}{2}}}$, which is the radius of curvature of the perpendicular to the meridian at the latitude l . If $l = 0$, $\sin l = 0$, and we have $R_m = a(1-e^2)$ and $R_n = a$. If $l = 90^\circ$, $\sin l = 1$, and we have $R_m = R_n = \frac{a}{(1-e^2)^{\frac{1}{2}}}$. In the former case the normal ellipse BL coincides with the equator, while in the latter it coincides with the meridian at the pole.

The radius of curvature of any normal ellipse on the spheroid, whether coinciding with the plane of the meridian, at right angles, or oblique to it, may also be obtained in the following manner. The radius of curvature of the meridian may be derived, at once, from the equation of the meridional ellipse $y^2 = (1-e^2)(a^2 - x^2)$, by substituting the values of the first and second differential coefficients in the general expression for the radius of the osculating circle, and making the proper substitutions and eliminations. It will be found that

$$\frac{dy}{dx} = -\frac{(1-e^2)x}{a^2 - x^2}, \quad \text{and} \quad \frac{d^2 y}{dx^2} = -\frac{a^2(1-e^2)}{(a^2 - x^2)^{\frac{3}{2}}},$$

whence
$$R_m = \frac{(a^2 - e^2 x^2)^{\frac{1}{2}}}{a^2 (1 - e^2)} = \frac{a (1 - e^2)}{(1 - e^2 \sin^2 l)^{\frac{3}{2}}}.$$

The radius of curvature of the perpendicular to the meridian is the normal LN produced to the axis of the spheroid, whence it is evident that $MN : CM :: LN : R_n$, or $S : x :: n : R_n$ and consequently

$$R_n = \frac{nx}{S} = \frac{nx}{(1 - e^2)x} = \frac{n}{1 - e^2} = \frac{a}{(1 - e^2 \sin^2 l)^{\frac{3}{2}}}.$$

It can then be easily proved that

$$R_z = \frac{R_m R_n}{R_m \sin^2 z + R_n \cos^2 z}^* = \frac{a (1 - e^2)}{(1 - e^2) (1 - e^2 \sin^2 l)^{\frac{3}{2}} \sin^2 z + (1 - e^2 \sin^2 l)^{\frac{3}{2}} \cos^2 z}.$$

In order to find the length of a small arc δ° of a normal ellipse, in any position, on the surface of the spheroid, we have merely to multiply the radius of curvature at l , the middle latitude of δ° , by π , the ratio of the semi-circumference to the radius of a circle, and divide the product by $\frac{180^\circ}{\delta^\circ}$. If $\delta = 1^\circ$, we have

$$D_n = \frac{\pi}{180} R_n = \frac{\pi}{180} \cdot \frac{a}{(1 - e^2 \sin^2 l)^{\frac{3}{2}}} = \frac{\pi a}{180} (1 - e^2 \sin^2 l)^{-\frac{3}{2}},$$

which is the length of an arc of one degree of the perpendicular to the meridian at the mean latitude l ;

$$D_m = \frac{\pi}{180} R_m = \frac{\pi}{180} \cdot \frac{a (1 - e^2)}{(1 - e^2 \sin^2 l)^{\frac{3}{2}}} = \frac{\pi a (1 - e^2)}{180} (1 - e^2 \sin^2 l)^{-\frac{3}{2}},$$

which is the length of an arc of one degree of the meridian at the mean latitude l ; and

$$D_z = \frac{\pi}{180} R_z = \frac{\pi}{180} \cdot \frac{R_m R_n}{R_m \sin^2 z + R_n \cos^2 z},$$

which is the length of an arc of one degree making any angle z with the meridian, at the mean latitude l .

* This Formula is wrong in Capt. T. J. LEE's Tables and Formulæ, Article XI. p. 34.

If, in the last expressions for the length of a degree of the perpendicular and meridian, we develop the second factors by the binomial theorem, we have

$$D_n = \frac{\pi a}{180} (1 + \frac{1}{2} e^2 \sin^2 l + \frac{3}{8} e^4 \sin^4 l + \frac{5}{16} e^6 \sin^6 l + \frac{35}{128} e^8 \sin^8 l + \&c.),$$

and

$$D_m = -\frac{\pi a (1-e^2)}{180} (1 + \frac{3}{2} e^2 \sin^2 l + \frac{15}{8} e^4 \sin^4 l + \frac{35}{16} e^6 \sin^6 l + \frac{315}{128} e^8 \sin^8 l + \&c.).$$

Now $\pi = 3.141592654$, and, according to the data adopted by the Superintendent of the United States Coast Survey, $a = 6377397.156$ metres, and $e = 0.081696830$. Substituting these values for π , a , and e , in the developed expressions for the length of a degree of the perpendicular and meridian, we have, in powers of $\sin l$,

$$\begin{aligned} (1) \quad D_n &= 111306.57808 + 371.45076 \sin^2 l + 1.85940 \sin^4 l \\ &\quad + 0.01034 \sin^6 l + 0.0006 \sin^8 l. \\ (2) \quad D_m &= 110563.67657 + 1106.91467 \sin^2 l + 9.23495 \sin^4 l \\ &\quad + 0.07191 \sin^6 l + 0.00054 \sin^8 l. \end{aligned}$$

$$\begin{aligned} \text{But} \quad \sin^2 l &= 1 - \cos^2 l; & \sin^4 l &= 1 - 2 \cos^2 l + \cos^4 l; \\ \sin^6 l &= 1 - 3 \cos^2 l + 3 \cos^4 l - \cos^6 l; \\ \text{and} \quad \sin^8 l &= 1 - 4 \cos^2 l + 6 \cos^4 l - 4 \cos^6 l + \cos^8 l. \end{aligned}$$

Substituting these values for the different powers of $\sin l$ in (1) and (2), we have, for the length of a degree of the perpendicular and meridian in powers of $\cos l$,

$$\begin{aligned} (3) \quad D_n &= 111679.89864 - 375.20082 \cos^2 l + 1.89079 \cos^4 l \\ &\quad - 0.01058 \cos^6 l + 0.00006 \cos^8 l. \\ (4) \quad D_m &= 111679.89864 - 1125.60246 \cos^2 l + 9.45392 \cos^4 l \\ &\quad - 0.07407 \cos^6 l + 0.00054 \cos^8 l. \end{aligned}$$

$$\begin{aligned} \text{Now} \quad \sin^2 l &= \frac{1}{2} - \frac{1}{2} \cos 2l; & \sin^4 l &= \frac{3}{8} - \frac{1}{2} \cos 2l + \frac{1}{8} \cos 4l; \\ \sin^6 l &= \frac{5}{16} - \frac{3}{2} \cos 2l + \frac{3}{8} \cos 4l - \frac{1}{32} \cos 6l; \end{aligned}$$

and $\sin^8 l = \frac{35}{128} - \frac{7}{16} \cos 2l + \frac{7}{32} \cos 4l - \frac{1}{16} \cos 6l + \frac{1}{128} \cos 8l$. Substituting these values of the different powers of $\sin l$ in (1) and (2), or changing all the negative into positive signs, which changes the values to those of the powers of $\cos l$ instead of $\sin l$, and substituting in (3) and (4), we have, for the length of a degree of the perpendicular and meridian in multiples of $\cos l$,

$$(5) \quad D_n = 111493.00398 - 186.65995 \cos 2l + 0.23438 \cos 4l - 0.00033 \cos 6l.$$

$$(6) \quad D_m = 111120.61963 - 558.10875 \cos 2l + 1.16797 \cos 4l - 0.00228 \cos 6l.*$$

If a perpendicular be let fall from the point L in the meridian PQ , upon the semi-axis CP , it will be the radius of the parallel of latitude at that point, and will obviously be equal to the abscissa CM . Hence we have $R_p = R_n \cos l$, the radius of the parallel of latitude, and consequently $D_p = D_n \cos l$, the length of a degree of the parallel at the latitude l . Therefore, multiplying (3) by $\cos l$, we have, for the length of a degree of the parallel at the latitude l in powers of $\cos l$,

$$(7) \quad D_p = 111675.89864 \cos l - 375.20082 \cos^3 l + 1.89079 \cos^5 l - 0.01058 \cos^7 l.$$

But $\cos^3 l = \frac{3}{4} \cos l + \frac{1}{4} \cos 3l$; $\cos^5 l = \frac{5}{8} \cos l + \frac{5}{16} \cos 3l + \frac{1}{16} \cos 5l$; and $\cos^7 l = \frac{35}{64} \cos l + \frac{21}{16} \cos 3l + \frac{7}{16} \cos 5l + \frac{1}{64} \cos 7l$. Substituting these values for the different powers of $\cos l$ in (7), we have, for the length of a degree of the parallel at the latitude l in multiples of $\cos l$,

$$(8) \quad D_p = 111399.67397 \cos l - 93.21281 \cos 3l + 0.11702 \cos 5l - 0.00017 \cos 7l.$$

In practice it will be necessary to retain only three decimal figures in these formulas, and it will be found more convenient to use the powers when we work by the logarithmic, and the multiples when we work by the natural cosines. The following are the formulas for

* The decimal part of the coefficient of $\cos 2l$ in this formula is wrong in the Report of the Superintendent of the United States Coast Survey, for the year 1853, p. 100.*

the powers and the multiples in French metres, American yards, and English yards.

I. IN POWERS OF $\cos l$

$$D_m = \begin{cases} 111679.899 - 1125.602 \cos^2 l + 9.454 \cos^4 l - 0.074 \cos^6 l, & \text{in F. metres.} \\ 122129.741 - 1230.925 \cos^2 l + 10.339 \cos^4 l - 0.081 \cos^6 l, & \text{in A. yards.} \\ 122136.829 - 1230.996 \cos^2 l + 10.339 \cos^4 l - 0.081 \cos^6 l, & \text{in E. yards.} \end{cases}$$

$$D_n = \begin{cases} 111679.899 - 375.201 \cos^2 l + 1.891 \cos^4 l - 0.011 \cos^6 l, & \text{in F. metres.} \\ 122129.741 - 410.308 \cos^2 l + 2.068 \cos^4 l - 0.012 \cos^6 l, & \text{in A. yards.} \\ 122136.829 - 410.332 \cos^2 l + 2.068 \cos^4 l - 0.012 \cos^6 l, & \text{in E. yards.} \end{cases}$$

$$D_p = \begin{cases} 111679.899 \cos l - 375.201 \cos^3 l + 1.891 \cos^5 l - 0.011 \cos^7 l, & \text{in F. metres.} \\ 122129.741 \cos l - 410.308 \cos^3 l + 2.068 \cos^5 l - 0.012 \cos^7 l, & \text{in A. yards.} \\ 122136.829 \cos l - 410.332 \cos^3 l + 2.068 \cos^5 l - 0.012 \cos^7 l, & \text{in E. yards.} \end{cases}$$

II. IN MULTIPLES OF $\cos l$

$$D_m = \begin{cases} 111120.620 - 558.109 \cos 2l + 1.168 \cos 4l - 0.002 \cos 6l, & \text{in F. metr.} \\ 121518.130 - 610.331 \cos 2l + 1.277 \cos 4l - 0.002 \cos 6l, & \text{in A. yds.} \\ 121525.183 - 610.366 \cos 2l + 1.277 \cos 4l - 0.002 \cos 6l, & \text{in E. yds.} \end{cases}$$

$$D_n = \begin{cases} 111493.004 - 186.660 \cos 2l + 0.234 \cos 4l, & \text{in French metres.} \\ 121925.359 - 204.126 \cos 2l + 0.256 \cos 4l, & \text{in American yards.} \\ 121932.434 - 204.137 \cos 2l + 0.256 \cos 4l, & \text{in English yards.} \end{cases}$$

$$D_p = \begin{cases} 111399.674 \cos l - 93.213 \cos 3l + 0.117 \cos 5l, & \text{in French metres.} \\ 121823.296 \cos l - 101.935 \cos 3l + 0.128 \cos 5l, & \text{in American yards.} \\ 121830.366 \cos l - 101.941 \cos 3l + 0.128 \cos 5l, & \text{in English yards.} \end{cases}$$

ON SPHERICAL ANALYSIS.

By GEORGE EASTWOOD, Saxonville, Mass.

[Continued from Page 304, Vol. II.]

PROPOSITION VI.

To find the intersection of two given great circles of the sphere.

In this proposition it is manifest that the abscissa, X , and the ordinate, Y , of the point of intersection are common to both the given circles. If therefore we define the equation of one of them by

$y = \tau x + \beta$, the equation of the other by $y = \tau' x + \beta'$, and the equation of their common point of intersection by

$$Y = \tau X + \beta = \tau' X + \beta',$$

we shall have
$$X = \frac{\beta' - \beta}{\tau - \tau'},$$

and
$$Y = \tau X + \beta = \frac{\tau \beta' - \tau \beta}{\tau - \tau'} + \beta = \frac{\tau \beta' - \tau' \beta}{\tau - \tau'},$$

for the point XY of intersection.

PROPOSITION VII.

To find the intersection of a great with a less circle of the sphere.

Let the centre of the less circle be taken for the origin of co-ordinates; and let, as in Cor. 2 of Prop. II., its equation be represented by

$$y^2 + 2xy \cos \omega + x^2 = r^2.$$

If we solve this equation for y , we shall find

$$y = (r^2 - x^2 \sin^2 \omega)^{\frac{1}{2}} - x \cos \omega.$$

But by (19)
$$y = \tau x + \beta,$$

$$\therefore r^2 - x^2 \sin^2 \omega = (\tau x + x \cos \omega + \beta)^2,$$

$$\therefore r^2 - \beta^2 = (\tau^2 + 2\tau \cos \omega + 1)x^2 + 2\beta(\tau + \cos \omega)x,$$

$$\therefore x = \frac{(r^2 - \beta^2)(\tau^2 + 2\tau \cos \omega + 1) - \beta^2 \sin^2 \omega}{\tau^2 + 2\tau \cos \omega + 1} - \beta(\tau + \cos \omega) = X,$$

$$\therefore y = \tau x + \beta = \frac{\tau(r^2 - \beta^2)(\tau^2 + 2\tau \cos \omega + 1) - \beta^2 \sin^2 \omega}{\tau^2 + 2\tau \cos \omega + 1} + \beta(\tau + \cos \omega) = Y.$$

Now, that this value of y may be possible, it is necessary that we have

$$r^2 > \frac{\beta^2 \sin^2 \omega}{\tau^2 + 2\tau \cos \omega + 1};$$

and that the two circles may touch each other, or be tangential, it is necessary and sufficient that

$$(27) \quad r = \pm \left(\frac{\beta^2 \sin^2 \omega}{\tau^2 + 2\tau \cos \omega + 1} \right)^{\frac{1}{2}} = \frac{\pm \beta \sin \omega}{(\tau^2 + 2\tau \cos \omega + 1)^{\frac{1}{2}}}.$$

As r , in this case, is equal to the distance between the pole of the less circle and the circumference of the great circle, it follows that (27) makes known by its tangent the distance from the origin of a great circle whose intercepts on the axes of reference are α and β .

If the axes of reference are rectangular, the condition of contact will become $\beta = \pm r(1 + \tau^2)^{\frac{1}{2}}$; so that the equation of a great circle of the sphere tangent to a less circle may be written

$$(28) \quad y = \tau x \pm r(1 + \tau^2)^{\frac{1}{2}}.$$

Equation (28) may be applied to the solution of the following problem : —

A less circle of the sphere has its centre at the right angle of a right spherical triangle; it is required to draw a great circle tangent to it, so that the tangents of the intercepted segments of the sides of the triangle may have a given ratio.

There are evidently two solutions to this problem: for if we seek the intersection of the two circles $y = \tau x \pm r(1 + \tau^2)^{\frac{1}{2}}$, we see at once that x and y cannot be finite for the two equations

$$y = \tau x + r(1 + \tau^2)^{\frac{1}{2}}, \quad y = \tau x - r(1 + \tau^2)^{\frac{1}{2}},$$

when they subsist together. That they may subsist, we must obviously have $x = \infty$; that is, the intersection must be 90° from the pole of the less circle. Hence, in this case, we must have recourse to the latitude of the point of intersection.

Take $\bar{y}$ for the latitude of the point of intersection, then by (5) $y = \frac{\bar{y}}{\cos x}$; so that the equations of the two tangent circles may be written

$$\bar{y} = \tau \sin x + r(1 + \tau^2)^{\frac{1}{2}} \cos x, \quad \bar{y} = \tau \sin x - r(1 + \tau^2)^{\frac{1}{2}} \cos x,$$

from which we readily deduce $\cos x = 0$, or $x = \frac{\pi}{2}$, and $\therefore \bar{y} = \tau$.

As a further illustration of the subject, let P , in the annexed

diagram, be the point of intersection of the two tangent circles, C and D their points of contact with the given circle, and AO, OB the intercepted segments of the sides of the given triangle. Then, because $OP = OE$ is a quadrantal arc of a great circle, and the angle

$$POC = \angle POD = \frac{\pi}{2},$$

we have the latitude

$$PE = \angle POE = \frac{\pi}{2} - \angle DOE = \frac{\pi}{2} - \angle AOC.$$

And, because the angles AOB, ACO, BCO are each right angles, the spherical right triangles ACO, BCO give

$$\tan CO = \tan AO \cdot \cos AOC = \tan BO \cdot \cos BOC = \tan BO \cdot \sin AOC;$$

$$\therefore \cot AOC = \frac{\tan BO}{\tan AO} = \frac{\beta}{\alpha} = \tau = \tan PE.$$

If $x'y'$ be the co-ordinates of the point of contact C or D , the equation of the tangent AP or PD may be written

$$(29) \quad yy' + xx' = r^2.$$

For, by (28), the equation of the point of contact is

$$y' = \tau x' + r^2(1 + \tau^2)^{\frac{1}{2}},$$

in which $r = (x^2 + y^2)^{\frac{1}{2}}$.

Resolving therefore for τ , we find

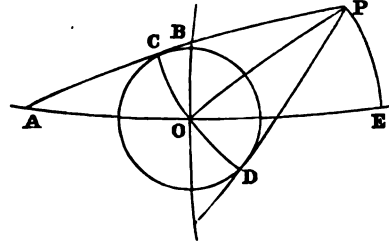
$$\tau = -\frac{x'}{y'} = \frac{-1}{\frac{y'}{x}} = -\frac{\beta}{\alpha},$$

the substitution of which in (28) gives $yy' + xx' = r^2$.

PROPOSITION VIII.

To find the distance between two given points on the surface of the sphere.

Designating the latitudes of the given points by $\bar{y}_1, \bar{y}_2$, the longi-



tudes by x' , x'' , and their distance apart by $\mathcal{A}$, we shall have from the fundamental spherical triangle,

$$\begin{aligned}\cos \mathcal{A} &= \sin \bar{y}_1 \sin \bar{y}_2 + \cos \bar{y}_1 \cos \bar{y}_2 \cos (x' - x''), \\ &= \frac{\bar{y}_1 \bar{y}_2 + \cos x' \cos x'' + \sin x' \sin x''}{(1 + \bar{y}^2)^{\frac{1}{2}} (1 + \bar{y}'^2)^{\frac{1}{2}}};\end{aligned}$$

in which $\sin \bar{y}_1 \sin \bar{y}_2$ and $\cos \bar{y}_1 \cos \bar{y}_2$ are replaced by their conventional equivalents $\bar{y}_1 \bar{y}_2$ and $(1 + \bar{y}_1^2)^{\frac{1}{2}} (1 + \bar{y}_2^2)^{\frac{1}{2}}$.

If now we introduce, for the geographical co-ordinates $\bar{y}_1$ and $\bar{y}_2$, the geometrical co-ordinates $y' \cos x'$ and $y'' \cos x''$, or their equals, $\frac{y'}{(1 - x'^2)^{\frac{1}{2}}}$ and $\frac{y''}{(1 - x''^2)^{\frac{1}{2}}}$, we shall have

$$(30) \quad \cos \mathcal{A} = \frac{y' y'' + x' x'' + 1}{\pm (1 + x'^2 + y'^2)^{\frac{1}{2}} (1 + x''^2 + y''^2)^{\frac{1}{2}}},$$

$$(31) \quad \sin \mathcal{A} = \frac{\{(x'' - x')^2 + (y'' - y')^2 + (x' y'' - x'' y')^2\}^{\frac{1}{2}}}{(1 + x'^2 + y'^2)^{\frac{1}{2}} (1 + x''^2 + y''^2)^{\frac{1}{2}}},$$

$$(32) \quad \tan \mathcal{A} = \frac{\pm \{(x'' - x')^2 + (y'' - y')^2 + (x' y'' - x'' y')^2\}^{\frac{1}{2}}}{y' y'' + x' x'' + 1}.$$

In these formulas the double signs indicate the distance of the point $x' y'$ from the point diametrically opposite to the point $x'' y''$, as well as from the point $x'' y''$ itself. This will be obvious when it is remembered that two points on the surface of the sphere may be joined by two different arcs of great circles, which, taken together, make up the whole circumference of a great circle.

If in equation (32) we regard $\mathcal{A}$ as constant, and x', y' , as variables, we shall have the equation of a less circle of the sphere referred to an external origin, in functions of the co-ordinates x'', y'' , of its pole, and of its radius $\mathcal{A}$.

When two points on the surface of the sphere are at a distance of 90° from each other, the cosine of this distance is zero; we shall therefore have, between their co-ordinates x', y', x'', y'' , the relation

$$(33) \quad y' y'' + x' x'' + 1 = 0.$$

When two great circles intersect each other at right angles, their poles will be 90° apart; hence the relation (33) will hold for the co-ordinates of their poles.

If two great circles are represented by the equations

$$Mx + Ny = P, \quad M'x + N'y = P',$$

the relation (33) will give

$$(34) \quad MM' + NN' + PP' = 0.$$

And, generally, if two great circles be defined by the equations

$$\frac{y}{\beta} + \frac{x}{\alpha} = 1, \quad \frac{y}{\beta'} + \frac{x}{\alpha'} = 1;$$

the relation which expresses that they cut each other at right angles will be truly defined by

$$(35) \quad \frac{1}{\beta\beta'} + \frac{1}{\alpha\alpha'} + 1 = 0.$$

PROPOSITION IX.

To find an expression for the angle formed by the intersection of two great circles of the sphere.

The angle formed by the intersection of the two circles is evidently equal to the distance between their poles; $\therefore$ if θ represents this distance, and if x', y', x'', y'' , be the co-ordinates of the poles, equation (30) will give

$$\cos \theta = \frac{y'y'' + x'x'' + 1}{(1 + x'^2 + y'^2)^{\frac{1}{2}} (1 + x''^2 + y''^2)^{\frac{1}{2}}},$$

for the cosine of the required angle.

But if the circles be represented by $\alpha y + \beta x = \alpha\beta$ and $\alpha'y + \beta'x = \alpha'\beta'$, Prop. IV., gives

$$x' = -\frac{1}{\alpha}, \quad y' = -\frac{1}{\beta} \quad \text{and} \quad x'' = -\frac{1}{\alpha'}, \quad y'' = -\frac{1}{\beta'}.$$

Hence

$$(36) \quad \cos \theta = \frac{\alpha\alpha' + \beta\beta' + \alpha\alpha'\beta\beta'}{(\alpha^2 + \beta^2 + \alpha^2\beta^2)^{\frac{1}{2}} (\alpha'^2 + \beta'^2 + \alpha'^2\beta'^2)^{\frac{1}{2}}}.$$

Equation (36) shows that if

$$(37) \quad \alpha \alpha' + \beta \beta' + \alpha \alpha' \beta \beta' = 0,$$

the two circles will be perpendicular to each other.

PROPOSITION X.

To find the distance of a given point on the surface of the sphere from the circumference of a great circle.

Let the given point be designated by $x'y'$, the pole of the given circle by $x''y''$, the distance between them by Δ , and the required distance by Δ' ; then, by (32),

$$\tan \Delta = \frac{\pm \{(x'' - x')^2 + (y'' - y')^2 + (x'y'' - x''y')^2\}^{\frac{1}{2}}}{y'y'' + x'x'' + 1}.$$

But $\tan \Delta' = \cot \Delta$, hence

$$(38) \quad \tan \Delta' = \frac{y'y'' + x'x'' - 1}{\pm \{(x'' - x')^2 + (y'' - y')^2 + (x'y'' - x''y')^2\}^{\frac{1}{2}}},$$

makes known the required distance.

If for x'' and y'' we substitute their values, $-\frac{1}{\alpha'}$ and $-\frac{1}{\beta'}$, then (38) becomes

$$(39) \quad \tan \Delta' = \frac{-\alpha'y' - \beta'x' + \alpha'\beta'}{\pm \{(\alpha'x' - 1)^2 + (\beta'y' - 1)^2 + (\alpha'x' - \beta'y')^2\}^{\frac{1}{2}}},$$

Again, since

$$\sin \Delta = \frac{\tan \Delta'}{(1 + \tan^2 \Delta')^{\frac{1}{2}}} = \frac{-\alpha'y' - \beta'x' + \alpha'\beta'}{\pm \{(\alpha'^2 + \beta'^2 + \alpha'^2\beta'^2)(1 + x'^2 + y'^2)\}^{\frac{1}{2}}},$$

$\therefore$ in general, if the equation of a great circle be represented by $\mu y + \nu x + \rho = 0$, then a perpendicular upon its circumference from a given point, $x'y'$, may be determined from the equation

$$(40) \quad \sin \Delta = \frac{\mu y + \nu x + \rho}{\mp \{(\mu^2 + \nu^2 + \rho^2)(1 + x'^2 + y'^2)\}^{\frac{1}{2}}}.$$

PROPOSITION XI.

To find the equation of a great circle passing through a given point, and perpendicular to another given great circle.

Let the given great circle be defined by the equation

$$(a) \quad \frac{y'}{\beta'} + \frac{x'}{\alpha'} - 1 = 0,$$

and the required one by the equation

$$(b) \quad \frac{y}{\beta} + \frac{x}{\alpha} - 1 = 0.$$

Then, that the conditions of the problem may be fulfilled, we must have, by (35) and (23),

$$(c) \quad \frac{1}{\alpha \alpha'} + \frac{1}{\beta \beta'} + 1 = 0, \quad y - y' = \tau' (x - x') = -\frac{\beta'}{\alpha'} (x - x').$$

Now, from (b) we find

$$(d) \quad \alpha = \frac{\beta x}{\beta - y},$$

and from (a) we find $\alpha' = \frac{\beta' x'}{\beta' - y'};$

$$(e) \quad \therefore \alpha \alpha' = \frac{\beta \beta' x x'}{(\beta - y) (\beta' - y')}.$$

Hence (c) becomes $\frac{(\beta - y) (\beta' - y')}{\beta \beta' x x'} + \frac{1}{\beta \beta'} + 1 = 0,$

which gives $\beta' = \frac{(\beta - y) y' - x' x}{\beta - y + \beta x' x} = \frac{\beta y' - \alpha x'}{(1 + \alpha x') \beta},$

by virtue of (d).

Again, from (e) we have

$$\alpha' = \frac{\beta \beta' x x'}{(\beta - y) (\beta' - y') \alpha} = \frac{\{(\beta - y) y' - x' x\} \beta}{-(\beta - y) (1 + \beta y') \alpha} = \frac{\beta y' - \alpha x'}{-(1 + \beta y') \alpha}.$$

Hence $\tau' = -\frac{\beta'}{\alpha'} = \frac{\alpha (1 + \beta y')}{\beta (1 + \alpha x')}$, and therefore the required equation is

$$(41) \quad y - y' = \frac{\alpha (1 + \beta y')}{\beta (1 + \alpha x')} (x - x').$$

If the co-ordinates of the pole of the given circle be x'' and y'' ,

then, since $\alpha = -\frac{1}{x'}$ and $\beta = -\frac{1}{y'}$, equation (41) gives

$$y - y' = \frac{y' - y''}{x' - x''} (x - x'),$$

which accords with (25), as it evidently ought to do.

A TREATISE ON DETERMINANTS.

CHAPTER I.

INTRODUCTORY.

1. If, by the usual methods of elimination, we solve the linear equations $a'x + b'y = m'$, $a''x + b''y = m''$, we get

$$x = \frac{m'b'' - m''b'}{a'b'' - a''b'}, \quad y = \frac{a'm'' - a''m'}{a'b'' - a''b'};$$

whose numerators and common denominator have each that peculiar "alternate symmetry," whereby either changes merely in sign if we transpose any two letters throughout without moving the indices; e. g. $(m'b'' - m''b') = -(b'm'' - b''m')$. So also, if we transpose two indices throughout without moving the letters; e. g. $(m'b'' - m''b') = -(m''b' - m'b'')$. Indeed the symmetry of either polynomial is precisely the same with respect to its indices as with respect to its letters. Moreover, the numerator of x or of y is formed from the common denominator, by writing m instead of a or instead of b .

All these properties belong as well to any n values x, y , &c., determined by n given linear equations. Thus,

$$\begin{aligned} x &= \frac{m'b''c''' + m''b'''c' + m'''b'c'' - m'''b''c' - m''b'c''' - m'b'''c''}{a'b''c''' + a''b'''c' + a'''b'c'' - a'''b''c' - a''b'c''' - a'b'''c''}, \\ y &= \frac{a'm''c''' + a''m'''c' + a'''m'c'' - a'''m''c' - a''m'c''' - a'm'''c''}{a'b''c''' + a''b'''c' + a'''b'c'' - a'''b''c' - a''b'c''' - a'b'''c''}, \\ z &= \frac{a'b''m''' + a''b'''m' + a'''b'm'' - a'''b''m' - a''b'm''' - a'b'''m''}{a'b''c''' + a''b'''c' + a'''b'c'' - a'''b''c' - a''b'c''' - a'b'''c''}, \end{aligned}$$

which evidently satisfy the equations

$$a'x + b'y + c'z = m', \quad a''x + b''y + c''z = m'', \quad a'''x + b'''y + c'''z = m''',$$

have these properties.

Again, the numerators and denominator of the set of values obtained from n equations may be found, by a simple law, from the values that satisfy sets of $(n-1)$ equations. Thus the above common denominator of x, y, z , may be written

$$\begin{aligned} & a'(b''c''' - b'''c'') + a''(b'''c' - b'c''') + a'''(b'c'' - b''c'), \\ \text{or} \quad & a'(b''c''' - c''b''') + b'(c''a''' - a''c''') + c'(a''b''' - b''a'''); \end{aligned}$$

whose parentheses are the denominators of those values that satisfy the equations

$$b'\xi + c'\eta = m', \quad b''\xi + c''\eta = m'', \quad b'''\xi + c'''\eta = m''';$$

or else the equations

$$a''\xi + a'''\eta = A, \quad b''\xi + b'''\eta = B, \quad c''\xi + c'''\eta = C,$$

taken two and two.

When the common denominator vanishes, some or all of $x, y, \dots$ must become infinite, indicating that the given equations are incompatible; unless all $m', m'', \dots$ likewise vanish, in which case $x, y, \dots$ are indeterminate, indicating that some of the given equations are deducible from the rest.

2. These numerators and denominators are called DETERMINANTS. Their symmetry should appear in most investigations where systems of linear equations enter; and their vanishing commonly indicates that certain co-efficients or functions are not independent. Their properties will often aid us when the difficult or characteristic feature of a problem is to eliminate among the equations or to combine the results of elimination.

The subject of Derivatives assumes a linear character, [we may

say, through the systematic omission of the higher infinitesimals,] such that the change of systems of independent variables in differentiating or integrating, the assignment of multiple definite integrals, and the properties of linear differential equations, so important in dynamics, involve determinants.

In Algebra, equations of any degree have been treated as linear equations, whose new variables are powers and products of the old ones, and may be eliminated by determinants. By such elimination among equations derived from a single one by differentiation, we obtain determinants; e. g., the “discriminant” whose vanishing shall express certain equalities among the roots, and which are “invariants.” The extensive theory of invariants, &c., has also other connections with determinants.

In Geometry, we naturally work by combining right lines and planes, whose equations are linear. Co-ordinates are often changed by “linear substitution,” as in Perspective, and this introduces Invariants, &c. The contents of triangles and pyramids, being definite integrals, are expressed by determinants.

In the Theory of Numbers and elsewhere, it is natural that those determinants whose vanishing changes the character of any problem, should likewise often affect it by their form or smallness when they do *not* vanish. Especially, the discriminant of a quadratic form so *determines* the character of that form, as to have first received from GAUSS the name “determinant.” (Disq. Arith. § ...)

The preference of linear forms, even in Geometry and by Nature in Dynamics, appears to be not merely because linear equations are the simpler, as quadratics are simpler than cubics; but also because, unlike all higher equations, they give but one value to each variable, which is always real if the co-efficients and absolute terms are so, the degree, too, being never raised by eliminating any of the variables.

3. Though regarded chiefly as a rather simple instrument of investigation, Determinants have also an interest of their own. I shall present what I suppose to be their leading features and their general modes of application.*

What follows will be chiefly from other treatises ; most from BRIOSCHI, but much also from SALMON, PEIRCE, FAA DE BRUNO, SPOTTISWOODE, and from several memoirs of CAYLEY and others ; with occasional suggestions of my own. Writing rather of the theory than of its authors, I could not always mention the latter without more reading, neither do I care to keep their work unchanged.

I shall try to state clearly, in coarse type, what is at first most important ; giving more concisely, and in finer print, things which the reader may omit at his option, as the fine print will never be needed to demonstrate anything that follows in coarse. But he would best notice any *examples* in small type which follow and illustrate the large.

Whether for mere *reference*, or to indicate final *results*, formulas, and perhaps verbal statements, will be numbered in one series, thus, (1), (2), . . . Some important results may be marked by bold figures, thus, (2). Analogous statements may be indicated by accents, as, (2), (2'), (2''). Subscript indices show which members of an equation are referred to ; e. g., (2_3) denotes the third member of (2), and (2_{125}) denotes the equality $(2_1) = (2_3) = (2_5)$.

* For further and instructive examples, see COMBESCURE's French translation of the admirable "Theory of Determinants and their principal Applications," by Prof. BRIOSCHI, of Pavia, 1856, 8vo, pp. 216 ; for the best elementary exposition of Invariants, &c., see SALMON's Higher Algebra, Dublin, 1859, 8vo, pp. 147 ; for a condensed and able account of Determinants as applied to Differential Equations, see PEIRCE's Analytic Mechanics, 1855, Chapter X. ; † and for the "General Theory of Elimination," see FAA DE BRUNO's work, Paris, 1859, pp. 224 ; SPOTTISWOODE's *Elementary Theorems relating to Determinants*, 1851, 4to, pp. 63.

† Almost the only other American notice of Determinants which I have seen, is an ingenious and suggestive article "On the Resolution of Symmetrical Equations," by Mr. STOCKWELL, *Astronomical Journal*, VI, 145, 1860.

I have given much thought to the determinant-notation, and especially to that of Chapter III. I have sought to select it from existing systems; but, as the best of these do not perfectly harmonize, I have ventured upon some changes, which, if found conducive to symmetry and convenience, will, I hope, be adopted by others.

Notation should be clearly separated into "temporary," adopted but for a single essay or part of one, and requiring redefinition if ever used again; and "permanent," to be used in the same sense in all memoirs by all writers who adopt it, even without redefinition, — yet always changeable *temporarily and by express definition*. But one notation may be permanently used in different senses, if they differ so *widely* or pertain to such different *subjects* as not to interfere. Of course, I have been most careful with such *permanent* notation as I *have not taken from others*.

Further explanations may be given in text or foot-notes where they are needed. Let us now commence our subject again.

[To be Continued.]

Mathematical Monthly Notices.

Concise Mathematical Operations; being a Sequel to the Author's Class-Books. With much additional Matter. By H. N. ROBINSON, LL.D., formerly Professor of Mathematics in the United States Navy, &c. New York: Ivison and Phinney. 1854.

Elements of Analytical Geometry, and the Differential and Integral Calculus. By H. N. ROBINSON, LL.D., Author of a Course of Mathematics, &c. New York: Ivison and Phinney, 48 and 50 Walker Street.

We include these two volumes in the same notice, because in the former, the author first presented his peculiar views of the Differential Calculus, about which we are at present mainly concerned. The latter work, as we learn from the publisher's advertisement, is now undergoing revision, and the two subjects which it contains will be published in separate volumes in January, 1861. As this will give the author a good opportunity to make such changes and improvements as the use of the work may have suggested, we beg to call his attention to a few features of the present edition, feeling desirous of doing all in our power to improve the text-

books from which our students get their first ideas of the science. In the Preface to the work on Mathematical Operations, the author says:—

"There has been a great deal of unnecessary controversy about the Differential and Integral Calculus, which we think can and ought to be wiped away. And we have here given a little foretaste of what we shall attempt if circumstances prompt us to write a work on that subject.

"It is not for us to assume that we can make science clearer than others, but we have yet to see the works of an author who has made the least attempt to show the simple elementary nature of this science. They at once commence with the definition of constants and variables, and then direct what to do.

"We have yet to see the first book that expends a word in giving an idea of what the Calculus is, or what is the utility and object of the science, and we charge more than half the obscurity to this fact alone: hence we could not forbear being a little elementary when we came to that subject, and we leave it to those readers, who have formerly studied other works on this science, to say whether we have or can dispel any of the obscurity that has so long hovered around it."

Again, on page 259:—

"One reason for the appearance of this work is that it is required, because able mathematicians have written so obscurely. They seem to have written as I should, were I indifferent whether the reader, or rather the learner, understood me or not."

We propose now to allow the author to state his own case; and then we can judge whether the "unnecessary controversy" has been "wiped away," either in the "little foretaste" where "we could not forbear being a little elementary," or in the work, which "circumstances" prompted.

The assurance with which the subject is approached, and the estimate which is put upon the labors of those who founded the science, would naturally lead the reader of the above passages to anticipate either great success or great failure, according to his knowledge of the subject and his acquaintance with the labors of the great masters of the science. We shall quote the exact language, italics, and punctuation of the author; and first from the former work:—

"The *differential calculus* is the science of minute variations, or of corresponding *small differences*—a science which owes its birth to the varying elements of Astronomy."—p. 302.

One of the author's correspondents asked him why he called a certain explanation an application of the differential calculus, to which he replied as follows:—

"If the operation I sent you is not calculation, I know not what it is—it may therefore be called calculus; and if in any operation small quantities may be omitted on account of their insignificance in relation to larger quantities, the small difference so omitted constitutes the *differential calculus*, and to obtain that general expression, you will see, by looking on page 193 of the Astronomy, that the powers of r above the first were omitted."—p. 314.

Arithmetic may therefore be called calculus; and because we omit decimal quantities, which on account of their smallness do not affect the required accuracy of the result, it is the differential calculus.

"The differential calculus is a branch of Analytical Geometry. It is a science for computing the *ratio* of small differences."—p. 315.

After giving the differentials of the circular functions, the author says:—

"All can understand how these equations are obtained; but what if we can? says the inquiring student. What use are they? What do we learn by them? It is useless to answer these questions by words only, we must show the answer by the following examples. Because the differential of a logarithm is the differential of the quantity divided by the quantity, therefore $d \log \sin x = \cot x dx$. For example, if we assume $x = 25^\circ$, and also assume $dx = 1'$, the differential, or difference between the $\log \sin 25^\circ$ and the $\log \sin 25^\circ 1'$ is expressed by $m \cot 25^\circ \times 1'$. We might assume $dx = 2'$ as well as $1'$, without error as far as six places of decimals; but it would not do to assume $dx =$ any large number of minutes; hence the differential calculus must be applied with judgment. Those who are naturally more nice than wise are commonly prejudiced against this science, and such frequently say it is no science at all; however, their objections are of no consequence."—pp. 322, 323.

Is it possible that the author does not know the use of TAYLOR'S Theorem?

We shall now give a few extracts from the *Elements of the Calculus*. As a first example the

author takes the equation $y = ax + b$, and, calling h the increment of x , finds $y' - y = a h$. He then says:—

"The first member of this equation is obviously the *increment* of y , whatever be the value of h , and when h is *extremely* small in relation to x , (we will not say *infinitely* small, as that word puzzles, and it is unnecessary) then $y' - y$ is extremely small in relation to y and in that case we write dy in place of $y' - y$, and dx in the place of h ." — p. 146.

Here, then, we have the author's theory of the Calculus. About how small he supposes a differential to be we are not informed; at any rate it is unnecessary, it seems, so far as the logic of the subject is concerned, to consider them infinitesimals. Perhaps we are expected to use our judgment here as well as in the applications.

"The terms *fluxion*, *differential*, and *derivative*, all mean the same thing. There are cases, as in *derived polynomials* in algebra, that it would not do to call the derivative a differential, as the increment might be too large." — p. 146.

We might suppose, were it not for the unfortunate reference to derived polynomials in algebra, that this supposed equality of meaning assigned to the derivative was a slip of the pen. But, at any rate, we learn from this quotation that an increment may be "too large."

On pages 149 and 150 we find a reference to the *limiting ratio*, but then it is said to be between the function and its variable. But this simple reference to another theory seems little more than accidental; for when we come to the differentiation of fractions on page 155, the author has got back upon the doctrine of "extremely small (not to say infinitely small)."

Such is our author's theory of the differential calculus; which seems to us strikingly like the calculus of finite differences used as the differential calculus. Indeed, if we were asked, as we have been, to give a definition of our author's differential calculus, we should say that it is a method of finding a change in a function corresponding to a given change in the variable, provided the change in the variable be not too great; in other words, it is a rather coarse way of computing changes in the values of functions using only first differences, and the judgment as to how large we may assume those differences to be.

Next, let us examine some specimens of demonstration, in the light of the theory:—

"Sometimes we have $u = f(y)$, and $y = F(x)$, and require the differential coefficient between the function u and the variable x . For that purpose we first find $\frac{du}{dy}$ from one equation, and then $\frac{dy}{dx}$ from the other, and multiply them together, and we have $\frac{du}{dx}$ as required." — p. 154.

It seems, then, that $\frac{du}{dy}$ and $\frac{dy}{dx}$ are the same as two ordinary fractions, and their product must be reduced to its lowest terms to obtain the required result.

How it follows, that in general, dy , as an arbitrary increment, equals dy as a dependent increment, and especially so, if neither is infinitely small, we are not informed. Indeed it is not a self-evident proposition applicable to functions involving any number of variables, in any theory of the calculus; and the difficulty is simply disguised for the sake of an apparent simplicity, which will deceive none but very young students.

"Hence we observe that whatever value is given to the variable y , *Taylor's theorem*, and the *binomial theorem*, will give the same result. Hence, when one of these theorems *fail* the other *fails*, but we never apply the word *fail* to the binomial theorem, and it is not clear to us that such an expression should ever be applied to Taylor's theorem."

If it be not clear to the author, how will it be to the student? The remark that "the *failure* is in the hypothesis and not in the theorems," shows that the author was probably correct when he said, "it is not clear to us."

The true question is, Does TAYLOR'S Theorem, for any function whatever, fail to give the true development to any number of terms for a given hypothesis, when some other method would not fail?

On pages 199 – 202 the tangent, sub-tangent, &c. of plane curves are found by means of the Differential Calculus, and on page 202 we find the remark that, “the student will perceive that these results are here obtained far more easily than in analytical geometry, but we are indebted to analytical geometry for the primary equation of the curve.” But a note on the bottom of the same page says that, “It is important that the student should observe that this portion of the calculus is pure analytical geometry.”

The idea that the calculus is a branch of analytical geometry is also impressed upon the student on pages 209, 217, 220, 229, as well as in the Preface, apparently, without the faintest suspicion that, as a calculus of functions, with its fundamental principles and its logic, it is entirely independent of any of its applications.

“The differential of an increasing quantity is *positive* before arriving at the maximum *zero*, and *negative* afterwards as we perceive by merely inspecting the figure, and *this is a general principle*. In like manner the differential of a decreasing quantity is *minus* before it attains its minimum point, it is *zero* at that point, and *positive* after passing that point. Hence, if the second differential of a function is *minus*, it indicates that the first differential corresponds to a *maximum*, and if *plus* it indicates a *minimum*. The foregoing illustrations are too plain and practical to meet the entire approbation of some minds; therefore, we give the following as more general and abstract.”

Why “some minds” should object to correct geometrical illustrations, and what kind of minds they must be if they do, we cannot understand; but we can comprehend why the “more general and abstract” discussion should not meet the entire approbation of some minds; for a cruder one we never examined. Some of the important points are entirely omitted; such, for instance, as the case in which the same values of the variable reduce the second derivative, as well as the first, to zero. Nothing is said of the maxima and minima of functions of more than one variable.

But we must, at present, omit many other points in the volumes before us for want of space. We have given them, and especially the parts devoted to the Differential and Integral Calculus, a careful and thorough examination; and we are constrained to say that we have never had the misfortune to examine two works, so totally unfitted for text-books, both as regards the treatment of the subjects and the style in which they are written.

We shall take an early opportunity of calling attention to other works by the same author.

Objects and Plan of an Institute of Technology; including, I. A Society of Arts; II. A Museum of Arts; and III. A School of Industrial Science. Proposed to be established in Boston. Prepared by direction of the Committee of Associated Institutions of Science and Arts; and addressed to Manufacturers, Merchants, Mechanics, Agriculturists, and other friends of enlightened industry in the Commonwealth.

WILLIAM B. ROGERS, *Chairman*.

MARSHALL P. WILDER,	JAMES M. BEEBE,	B. S. ROTCH,
SAMUEL H. GOOKIN,	DR. S. CABOT, JR.,	J. D. PHILBRICK,
ALFRED ORDWAY,	G. W. PRATT,	GEORGE B. EMERSON,
M. D. ROSS,	AMOS BINNEY,	R. C. WATERSTON,
ALEX. H. RICE,	DR. S. KNEELAND, JR.,	ERASTUS B. BIGELOW,
E. S. TOBEY,	CHARLES L. FLINT,	CHARLES H. DALTON,

Committee.

The Plan contemplates the following Standing Committees. *First. Committees of Arts.* On Mineral Materials; on Organic Materials; on Tools and Instruments; on Machinery and Motive Powers; on Textile Manufactures; on Manufactures of Wood, Leather, Paper, India Rubber, &c.; on Manufactures of Pottery, Glass, and Precious Metals; on Chemical Products and Processes; on Household Economy, including Warming, Illumination, Water-Supply, Ven-

tilation, and the Preparation and Preservation of Food; on Engineering and Architecture; on Commerce, Navigation, and Inland Transport; on the Graphic and Fine Arts. *Second.* Committee on the Museum. *Third.* Committee on the School of Industrial Science. *Fourth.* Committee on Publications. A very well defined outline of the duties of each committee is given; subject to such changes as experience shall hereafter suggest.

The School of Industrial Science and Art, is to be arranged under the following departments: 1st. A School of Design; 2d. A School of Mathematics; 3d. A School of Physics; 4th. A School of Chemistry; 5th. A School of Geology. Such is the general outline of a most admirable plan, with the details so arranged as to form a symmetrical and systematic whole.

The Report begins by setting forth in a clear and forcible manner the need now felt for the establishment of such an Institution, and the practical value of the results which would flow from it. In the following opinion we heartily concur: "In New England, and especially in our own Commonwealth, the time has arrived when, as we believe, the interests of Commerce and the Arts, as well as of General Education, call for the most earnest co-operation of intelligent culture with industrial pursuits. Our success hitherto in the competitions of trade, manufactures, and the other productive Arts, has been the admitted result of the superior intelligence which has inspired our enterprise and guided our activity; but, to secure a steady prosperity in the midst of the busy inventions and rapidly expanding knowledge which mark these pursuits in the leading European nations, we feel that it has become indispensable for us to provide, at least as effectually as they have done, such facilities for practical knowledge, and for the intelligent guidance of enterprise and labor, as may make our progress commensurate, step by step, with the advances of scientific and practical discovery."

We regret that space forbids our giving a full statement of the arrangement of the proposed Museum, and School of Industrial Science and Art. But the whole plan is well matured, and eminently practical; and of the desirableness of establishing such an Institution in Boston, there can be no question. Still, admitting all that its most enthusiastic friends claim for it, can the idea be carried into execution? We believe it can and will be done. It will need wise counsellors, and unyielding friends, who will be satisfied with nothing less than the largest measure of success. But these it has the good fortune to possess in the persons of the Committee under whose direction this comprehensive Report has been prepared. It will also need the most ample means; but these, we are assured by those who know, and have a right to speak in its behalf, will not be wanting; and it only remains to be seen whether our State will add this last and crowning excellence to her unsurpassed educational system by granting the desired location, upon which private munificence will rear the Institution.

On a Chart and Diagram for Facilitating Great-Circle Sailing. By HUGH GODFRAY, M. A., St. John's College, Cambridge. (From the Transactions of the Cambridge Philosophical Society, Vol. X. Part II.)

On charts, constructed on the *Central or Gnomonic Projection*, great circles are represented by straight lines; and thus, in Great-Circle Sailing, such charts answer the same purpose that Mercator's Charts do in Rhumb Sailing. To see the ship's track it is only necessary to draw a straight line from the ship's place to her destination. The author says: "By the addition of a Course and Distance Diagram I have made this projection answer all the conditions of Great-Circle Sailing with as much, if not more, facility than Mercator's chart does for sailing on a Rhumb. The track is *seen* a straight line, and this being drawn, the various courses and the distances to be run upon each are obtained, as also the distance from the ship to her destination, by a mere inspection of the diagram; and the chart can be used like an ordinary one for pricking off the ship's place from day to day."

Accompanying the Chart and Course and Distance Diagram, we have received from the

author a *Time Azimuth Diagram*, with a short memoir in explanation of its object, use, and construction. By means of the Diagram the true azimuth can be obtained without calculation to within one eighth of a degree. The data required are the latitude, sun's declination and apparent time at ship. It is hardly possible that such simple and elegant graphic solutions of these problems should not soon find their way into general use.

On the Secular Variations and Mutual Relations of the Orbits of the Asteroids. By SIMON NEWCOMB, S. B., A. A. S. (From the Memoirs of the American Academy, New Series, Vol. VIII.) Cambridge: Welch, Bigelow, and Co., Printers to the University. 1860.

The particular object of this Memoir is to submit OLBER's hypothesis of the origin of the asteroids to as rigorous a test as is consistent with the nature of the problem. If this hypothesis be true, then there must have been, and still be, certain relations among the asteroid orbits; and the test consists in determining whether these relations do in fact exist. The author has arranged the discussion under the following heads:—

§ 1. *Computation of the rigorous expressions in terms of the times of the elements of the asteroids.*

This section is treated with great elegance and skill. All the terms in these expressions, depending upon the elements of the asteroids, are given in tables with the mean distance of the asteroid as an argument; so that other asteroids may, with but trifling labor, be added to the list of twenty-five treated in this memoir, as fast as their elements become sufficiently well known.

§ 2. *Of the possibility that the orbits of all the asteroids once intersected in a common point.*

Carrying back the elements of the asteroid orbits by means of the formulas in § 1, there is found no tendency to a common point of intersection of the orbits.

§ 3. *Have the elements of the asteroid orbits ever been materially affected by a resisting medium?* Not sensibly.

§ 4. *Of the relations among the mean distances, eccentricities, and inclinations of the orbits of the asteroids; and between their masses and the velocities with which they must have been projected, if OLBER's hypothesis be true.*

The actual relations are not found to be those which would exist if OLBER's hypothesis be true.

§ 5. *Of certain observed relations among the asteroids which are the necessary or probable result of known causes, and therefore throw no light on the origin of the asteroids.*

This last section is very interesting, and contains satisfactory solutions of several important questions; such as the unequal distribution of the nodes and perihelia of the orbits; the interlinking of certain orbits, &c.

This valuable memoir is by far the most successful discussion ever made to settle one of the most deeply interesting problems in astronomy, which has agitated the scientific world for the last half-century; and will reflect great credit upon the ability of its author.

Plane Trigonometry, crown 8vo, pp. 271; *Spherical Trigonometry*, crown 8vo, pp. 112. For the use of Colleges and Schools. With numerous examples. By I. TODHUNTER, M. A., Fellow and Assistant Tutor of St. John's College, Cambridge. Macmillan and Co.: Cambridge, and 23 Henrietta Street, Covent Garden, London. 1859.

We do not call the attention of teachers and students to these volumes because we think that there are no good text-books upon these subjects in this country. On the contrary, we have a few excellent ones; and we have no hesitation in saying that we know of no foreign work which surpasses PROFESSOR CHAUVENET's *Treatise on Plane and Spherical Trigonometry* in all the elements of a good text-book, except, perhaps, in its collection of examples for practice. In this respect these volumes are especially worthy the attention of teachers; the one on Plane Trigonometry contains over six hundred examples, and the other about one hundred and fifty. But they are not ordinary works in other respects. We are particularly interested in the

chapters on the Construction of Trigonometrical Tables; Theory of Proportional Parts; Expansions of Trigonometrical Functions; Summation of Trigonometrical Series; Resolution of Trigonometrical Expressions into Factors; Area of a Spherical Triangle. Spherical-Excess; Geodetical Operations.

The author, after showing the connection of the values of the old and new Trigonometric Functions, remarks that, "The old definitions give some indications of the origin of the terms *sine*, *cosine*, &c." The word *sine* seems derived from the Latin *sinus*, a bosom, the arc is supposed to represent a bow, and thus gets its name, and the string, half of which represents the sine of half the arc, would come against the breast of the archer. The words *tangent* and *secant* are naturally derived from the old definitions. (See *Penny Cyclopædia*, Art. *Trig.*)

The modern method has now completely superseded the ancient method in English works; it was introduced by DR. PEACOCK. (See PEACOCK'S *Algebra*, Vol. II. p. 151.) It may, however, be observed, that it is stated by PROFESSOR DE MORGAN (*Trig. and Double Algebra*, p. 18), that RHETICUS, who gave the first complete trigonometrical table, and invented the secant and cosecant to complete it, used the method of ratios."

We find in the *Spherical Trigonometry*, p. 38, the method of proof of NAPIER'S Rules indicated by NAPIER in his *Mirifici Logarithmorum Canonis Descriptio*, pp. 32, 35.

In this connection the author gives a collection of opinions with respect to the practical utility of NAPIER'S Rules. "In the whole compass of mathematical science there cannot be found, perhaps, rules which more completely attain that which is the proper object of rules, namely, facility and brevity of computation." WOODHOUSE'S *Trig.*, Chap. X. "In the opinion of DELAMBRE (and no one was better qualified by experience to give an opinion) these theorems are best recollected by the practical calculator in their unconnected form." (AIRY'S *Trig.*, *Encyclopædia Metropolitana*.) "There are certain mnemonical formulæ called NAPIER'S Rules of Circular Parts, which are generally explained. We do not give them, because we are convinced that they only create confusion instead of assisting the memory."

We had always supposed by dividing a circle into five parts, and writing upon them in order, $a, \frac{\pi}{2} - B, \frac{\pi}{2} - c, \frac{\pi}{2} - A, b$; then selecting any one of these five quantities, and observing that of the remaining four, two ($b, \frac{\pi}{2} - B$) are adjacent and two ($\frac{\pi}{2} - c, \frac{\pi}{2} - A$) opposite; that the rules

$$\begin{aligned} \sin a &= \tan b \tan \left(\frac{\pi}{2} - B \right) = \cos \left(\frac{\pi}{2} - c \right) \cos \left(\frac{\pi}{2} - A \right) \\ &= \tan b \cot B \quad \quad = \sin c \sin A, \end{aligned}$$

in which the mind associates *sine* and *middle*, *tangent* and *adjacent*, *cosine* and *opposite*, could not be forgotten or misapplied by the student. Such we should think, from the admirable and lucid manner in which the subject is presented, must be the author's opinion.

Our examination of these volumes confirms us in the high opinion we entertain of the author's series of text-books.

T H E

MATHEMATICAL MONTHLY.

Vol. III . . . JANUARY, 1861. . . . No. IV.

PRIZE PROBLEMS FOR STUDENTS.

I. $ABCD$ is a square, AC its diagonal, bisect AD in E , join BE cutting AC in F ; then will

$$\frac{\triangle AEF}{1} = \frac{\triangle CEF}{2} = \frac{\triangle ABE}{3} = \frac{\triangle BCF}{4}.$$

II. If $\frac{a-b}{ay+bx} = \frac{b-c}{bz+cx} = \frac{c-a}{cy+az} = \frac{a+b+c}{ax+by+cz}$; then each of these fractions $= \frac{1}{x+y+z}$.

III. If d be the perpendicular, dropped from the angle C on the opposite side of a triangle, show that $\sin C = \frac{cd}{ab}$. Deduce the formula

$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

IV. An hyperbola, having its asymptotes parallel to the axes of co-ordinates, may be made to pass through the angular points of the quadrilateral formed by the four lines whose equations are

$$\begin{array}{ll} (1) \ y + x + 6a = 0, & (3) \ y + 6x + a = 0, \\ (2) \ y + 2x + 3a = 0, & (4) \ y + 3x + 2a = 0. \end{array}$$

V. Three lines, x, y, z , are drawn from the angular points of a triangle to a point within it. Show that when $x^n + y^n + z^n$ is a minimum we have

$$\frac{\sin(x, y)}{x^{n-1}} = \frac{\sin(x, z)}{y^{n-1}} = \frac{\sin(y, z)}{z^{n-1}},$$

in which (x, y) denotes the angle between x and y .

Solutions of these problems must be received by March 1, 1861.

NOTES AND QUERIES.

1. *To express the sines and cosines of the sum and difference of two angles in terms of the sines and cosines of the angles themselves.*

Let O be the origin of rectangular co-ordinates, and the pole of polar co-ordinates, the axis of x being the polar axis; then the square of the distance between two points expressed in polar co-ordinates (r, φ) , (r', φ') , and in rectangular co-ordinates (x, y) , (x', y') , is

$$(1) \quad r^2 + r'^2 - 2rr' \cos(\varphi - \varphi') = (x - x')^2 + (y - y')^2.$$

But these two systems of co-ordinates are connected by the equations

$$\begin{aligned} r^2 &= x^2 + y^2, & x &= r \cos \varphi, & y &= r \sin \varphi, \\ r'^2 &= x'^2 + y'^2, & x' &= r' \cos \varphi', & y' &= r' \sin \varphi', \end{aligned}$$

and therefore (1) reduces to

$$(2) \quad \cos(\varphi - \varphi') = \frac{x}{r} \cdot \frac{x'}{r'} + \frac{y}{r} \cdot \frac{y'}{r'} = \cos \varphi \cos \varphi' + \sin \varphi \sin \varphi'.$$

If we change the sign of φ' , then $\sin \varphi'$, or y' , will change its sign; therefore

$$(3) \quad \cos(\varphi + \varphi') = \frac{x}{r} \cdot \frac{x'}{r'} - \frac{y}{r} \cdot \frac{y'}{r'} = \cos \varphi \cos \varphi' - \sin \varphi \sin \varphi'.$$

From (1) we get

$$rr' \cos(\varphi - \varphi') = xx' + yy';$$

$$\therefore r^2 r'^2 - r^2 r'^2 \cos^2(\varphi - \varphi') = r^2 r'^2 - (xx' + yy')^2,$$

$$\therefore r^2 r'^2 \sin^2(\varphi - \varphi') = x^2 y'^2 - 2xx'yy' + x'^2 y^2,$$

$$\therefore rr' \sin(\varphi - \varphi') = xy' - yx',$$

$$\therefore \sin(\varphi - \varphi') = \frac{x}{r} \cdot \frac{y'}{r'} - \frac{y}{r} \cdot \frac{x'}{r'} = \sin \varphi \cos \varphi' - \cos \varphi \sin \varphi',$$

and $\sin(\varphi + \varphi') = \frac{x}{r} \cdot \frac{y'}{r'} + \frac{y}{r} \cdot \frac{x'}{r'} = \sin \varphi \cos \varphi' + \cos \varphi \sin \varphi'.$

2. *In a spherical triangle, to express the cosine of an angle in terms of sines and cosines of the sides.*

Let ABC be any spherical triangle, O the centre of the sphere, being the origin of co-ordinates, and OC the axis of z ; let r denote the radius of the sphere, and $(x, y, z), (x', y', z')$ the co-ordinates of A and B respectively; then the square of the chord AB gives

$$(1) \quad r^2 + r^2 - 2r^2 \cos AOB = (x - x')^2 + (y - y')^2 + (z - z')^2.$$

But these two systems of co-ordinates are connected by the equations

$$\begin{aligned} r^2 &= x^2 + y^2 + z^2, & r^2 &= x'^2 + y'^2 + z'^2, \\ x &= r \sin \theta \cos \varphi, & x' &= r \sin \theta' \cos \varphi', \\ y &= r \sin \theta \sin \varphi, & y' &= r \sin \theta' \sin \varphi', \\ z &= r \cos \theta, & z' &= r \cos \theta'; \end{aligned}$$

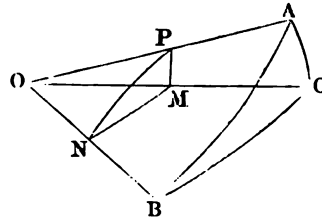
and (1) reduces to

$$\begin{aligned} \cos AOB &= \frac{x}{r} \cdot \frac{x'}{r} + \frac{y}{r} \cdot \frac{y'}{r} + \frac{z}{r} \cdot \frac{z'}{r} \\ &= \sin \theta \sin \theta' \cos \varphi \cos \varphi' + \sin \theta \sin \theta' \sin \varphi \sin \varphi' + \cos \theta \cos \theta' \\ &= \sin \theta \sin \theta' (\cos \varphi \cos \varphi' + \sin \varphi \sin \varphi') + \cos \theta \cos \theta' \\ &= \sin \theta \sin \theta' \cos (\varphi - \varphi') + \cos \theta \cos \theta'; \end{aligned}$$

$$\text{or} \quad \cos c = \sin a \sin b \cos C + \cos a \cos b.$$

The method of proof used in these notes is perfectly general, since all the equations used are universally true.

3. *Note on NAPIER'S Rules of Circular Parts.* — The formulas for the solution of right-angled spherical triangles are very easily deduced, as follows: Let the triangle ABC be right angled at C , and O be the centre of the sphere. Take any point, P , in AO ; draw PM perpendicular to OC , MN perpendicular to OB , and join NP . Since the planes AOC and BOC are perpendicular to each other,



$$\therefore PN^2 = PM^2 + MN^2 = OP^2 - OM^2 + OM^2 - ON^2 = OP^2 - ON^2;$$

therefore PNO is a right angle. Hence

- (1) $\frac{ON}{OP} = \frac{ON}{OM} \cdot \frac{OM}{OP}$; i. e. $\cos c = \cos a \cos b$.
- (2) $\frac{PM}{OP} = \frac{PM}{PN} \cdot \frac{PN}{OP}$; i. e. $\sin b = \sin B \sin c$, $\sin a = \sin A \sin c$.
- (3) $\frac{MN}{ON} = \frac{MN}{PN} \cdot \frac{PN}{ON}$; i. e. $\tan a = \cos B \tan c$, $\tan b = \cos A \tan c$.
- (4) $\frac{PM}{OM} = \frac{PM}{MN} \cdot \frac{MN}{OM}$; i. e. $\tan b = \tan B \sin a$, $\tan a = \tan A \sin b$.

The last equations in (2), (3), (4), follow by a simple interchange of letters.

From the product of the second and third equations of (4) we get

$$\tan A \tan B = \frac{\tan a \tan b}{\sin a \sin b} = \frac{1}{\cos a \cos b} = \frac{1}{\cos c} \text{ by (1).}$$

$$(5) \quad \therefore \cos c = \cot A \cot B.$$

Multiply crosswise the third equation of (2) and the second of (3); we obtain

$$\sin a \cos B \tan c = \tan a \sin A \sin c.$$

$$\therefore \cos B = \frac{\sin A \cos c}{\cos a} = \sin A \cos b \text{ by (1).}$$

$$(6) \quad \therefore \cos B = \sin A \cos b, \quad \cos A = \sin B \cos a.$$

Out of the five quantities a, b, c, A, B , only ten combinations of three can be formed; and, therefore, the ten equations just found solve all the cases of right-angled spherical triangles. Any two of the five quantities being given, and a third required, some one of these ten equations will give the required quantity in terms of the given ones.

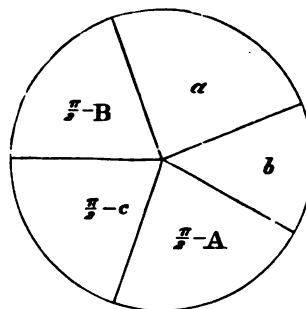
These ten equations are comprised in the six following corresponding theorems, since four of them apply twice to the same triangle.

- (1) $\cos \text{hyp} = \text{product of cosines of sides.}$
- (5) $\cos \text{hyp} = \text{products of cotangents of angles.}$

- (2) $\sin \text{ side} = \text{sine of opposite angle} \times \sin \text{ hyp.}$
- (3) $\tan \text{ side} = \tan \text{ hyp} \times \cos \text{ included angle.}$
- (4) $\tan \text{ side} = \tan \text{ opposite angle} \times \text{sine of other side.}$
- (6) $\cos \text{ angle} = \cos \text{ opposite side} \times \text{sine of other angle.}$

These six formulas may be condensed into two rules, known as NAPIER'S *Rules of Circular Parts*, which were first published by him in a work entitled *Mirifici Logarithmorum Canonis Descriptio*. Edinburgh, 1614.

Omitting the right angle, the circular parts are the two sides, including the right angle, and the complements of the hypotenuse and the two other angles; thus, $a, b, \frac{\pi}{2} - A, \frac{\pi}{2} - c, \frac{\pi}{2} - B$, are the five circular parts, and selecting any one of the five as the *middle part*, two of the remaining four are *adjacent*, and the other two *opposite* to the middle part. Then NAPIER'S two rules are



sine of middle part = product of tangents of adjacent parts.

sine of middle part = product of cosines of opposite parts.

These rules are memorized for use by associating *sine* and *middle*, *tangent* and *adjacent*, *cosine* and *opposite*.

The following table shows that these two rules involve the ten cases already found; the middle part stands first, the adjacent parts next, and the opposite parts in the third member of the equations.

$$\sin a = \tan b \tan \left(\frac{\pi}{2} - B \right) = \cos \left(\frac{\pi}{2} - A \right) \cos \left(\frac{\pi}{2} - c \right), \text{ by Rules ;}$$

$$\sin a = \tan b \cot B = \sin A \sin c. \quad \text{Reduced.}$$

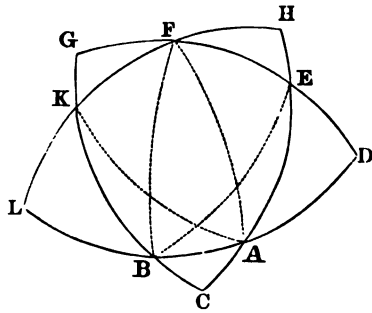
$$\sin b = \tan a \tan \left(\frac{\pi}{2} - A \right) = \cos \left(\frac{\pi}{2} - B \right) \cos \left(\frac{\pi}{2} - c \right), \text{ by Rules ;}$$

$$\sin b = \tan a \cot A = \sin B \sin c. \quad \text{Reduced.}$$

$$\begin{aligned}
 \sin\left(\frac{\pi}{2}-A\right) &= \tan b \tan\left(\frac{\pi}{2}-c\right) = \cos a \cos\left(\frac{\pi}{2}-B\right), & \text{by Rules;} \\
 \cos A &= \tan b \cot c &= \cos a \sin B. & \text{Reduced.} \\
 \sin\left(\frac{\pi}{2}-c\right) &= \tan\left(\frac{\pi}{2}-A\right) \tan\left(\frac{\pi}{2}-B\right) = \cos a \cos b, & \text{by Rules;} \\
 \cos c &= \cot A \cot B &= \cos a \cos b. & \text{Reduced.} \\
 \sin\left(\frac{\pi}{2}-B\right) &= \tan a \tan\left(\frac{\pi}{2}-c\right) = \cos b \cos\left(\frac{\pi}{2}-A\right), & \text{by Rules;} \\
 \cos B &= \tan a \cot c &= \cos b \sin A. & \text{Reduced.}
 \end{aligned}$$

Besides this method of proving these rules, NAPIER has indicated another, on pp. 32–35 of the work already referred to, which is as follows :—

Let ABC be a spherical triangle right-angled at C . With A and B as poles describe great circles, and produce all the sides of ABC



to meet them. The five triangles ABC , ADE , EHF , FGK , KLB are all right angled, and we propose to show that they all have the *same circular parts*. Let us first consider ABC and ADE . Denote the elements of ABC , as usual, by a, b, c, A, B ; then, in the triangle ADE , the angle $EAD = A$; the side AD is

the complement of AB , and equals $\frac{\pi}{2} - c$; the side DE equals $\frac{\pi}{2} - B$, because it measures the angle EBD , which is the complement of $ABC = B$; the angle DEA equals $\frac{\pi}{2} - a$, because it is the complement of the angle BEC , which is measured by $BC = a$; the side EA is the complement of AC , and equals $\frac{\pi}{2} - b$. The elements of ADE are therefore $\frac{\pi}{2} - c, \frac{\pi}{2} - B, \frac{\pi}{2} - b, \frac{\pi}{2} - a, A$, and its circular parts are $\frac{\pi}{2} - c, \frac{\pi}{2} - B, b, a, \frac{\pi}{2} - A$, which are those of ABC .

It can be shown in the same way that the triangles AED and EHF have the same circular parts, and so on.

The following table exhibits in one view all the elements of these five allied triangles, beginning with the hypotenuse, and numbering around to the right.

ABC	ADE	EHF	FGK	KLB
1. $AB=c$	$AE=\frac{\pi}{2}-b$	$FE=B$	$KF=A$	$BK=\frac{\pi}{2}-a$
2. $BAC=A$	$DEA=\frac{\pi}{2}-a$	$HEF=c$	$GKF=\frac{\pi}{2}-b$	$LBK=B$
3. $CA=b$	$DE=\frac{\pi}{2}-B$	$HF=\frac{\pi}{2}-A$	$GK=a$	$LB=\frac{\pi}{2}-c$
4. $BC=a$	$AD=\frac{\pi}{2}-c$	$EH=b$	$FG=\frac{\pi}{2}-B$	$KL=\frac{\pi}{2}-A$
5. $ABC=B$	$EAD=A$	$HEF=\frac{\pi}{2}-a$	$KFG=c$	$LKB=\frac{\pi}{2}-b$

The complements of the 3d, 4th, 5th, and 1st elements of any one of these triangles are the 1st, 2d, 3d, and 4th elements of the right-hand adjacent triangle, while their 2d and 5th elements are the same; and by applying this law to the five triangles in succession, we shall come back to the set of elements from which we started.

If we apply theorems (1) and (5) to each of these five triangles, or either of the remaining four theorems twice to each one of them, or all the theorems to any one of them, we shall obtain the ten original equations.

It follows, therefore, since the circular parts of these triangles are the same, that we obtain the same results by applying the two *Rules of Circular Parts* to each of the five circular parts of the same triangle, as by applying the theorems to the allied triangles, either all to one, or two to each of the five in succession.

This method considers each rule, not as the statement of dissimilar properties of one triangle, but as the statement of similar properties of five allied triangles. These notes on Spherical Trigonometry are substantially from TODHUNTER's excellent work on this subject.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE
PRIZE PROBLEMS IN No. XII., Vol. II.

The first Prize is awarded to ASHER B. EVANS, Madison Univ., N. Y.

The second prize is not awarded.

The third prize is awarded to JOHN M. TERRILL, Linneus, Mo.

PRIZE SOLUTION OF PROBLEM I.

By Miss H. S. HAZELTINE, Worcester, Mass.

If AB , CD be chords of a circle at right angles to each other, prove that the sum of the arcs AC , BD is equal to half the circumference.

Designate by E the point where the chords intersect each other, and draw the chord AC . In the right triangle AEC , the sum of the acute angles is 90° ; whence

$$\text{arc } BC + \text{arc } DD = \text{semi-circumference} = \text{arc } AC + \text{arc } BD,$$

or $180^\circ - (\text{arc } AC + \text{arc } BD)$, according as E falls outside or inside of the circle.

PRIZE SOLUTION OF PROBLEM II.

By JOHN M. TERRILL, Linneus, Mo.

Prove that the cube of any number and the number itself, being divided by 6, leave the same remainder.

Take $a = 6b + c$. Then $a^3 = 216b^3 + 108b^2c + 18bc^2 + c^3$. In dividing this last expression by 6, if there be a remainder it must be found in the last term, c^3 , because all the other terms are multiples of 6. This remainder is found by trial to be equal to c for all values of the latter less than 6, that is, for all possible values of c ; that is, when c is 1, 2, 3, 4, or 5, the remainders of c^3 divided by 6 are 1, 2, 3, 4, 5.

SECOND SOLUTION.

By Miss MARY A. WILLIS, Nunda, N. Y.

We know (MATH. MONTHLY, Vol. II., No. XI.) that $x^3 - x$ is divisible by 6 when x is a whole number. Therefore, since $x^3 = 6q + r$

and $x = 6q' + r'$, in which q, q' are the quotients, and r, r' the remainders, the equation

$$x^3 - x = 6(q - q') + r - r'$$

must be divisible by 6; and therefore $r - r'$ is divisible by 6, which cannot be the case, since $r - r' < 6$. Therefore $r - r' = 0$, or $r = r'$, as was to be proved.

PRIZE SOLUTION OF PROBLEM III.

By ASHER B. EVANS, Madison University, Hamilton, N. Y.

It is required to divide a given right cone into two parts, having the ratio of m to n , by a plane making an angle of p degrees with the axis.

Draw FG and DE parallel to the base BC and cutting the axis of the cone in H and I ; draw AL perpendicular to DG produced, and join H and L ; then, if MN be the diameter, conjugate to DG in the elliptical section $GMDN$, MN will be equal to $\sqrt{(DE \cdot FG)}$. The part ADG , above the elliptical section is an oblique elliptical cone, and

$$\frac{AL}{3} \cdot DG \cdot MN \frac{\pi}{4} = \frac{\pi}{12} AL \cdot DG \sqrt{(DE \cdot FG)} = \text{its solidity.}$$

The solidity of the whole cone ABC is

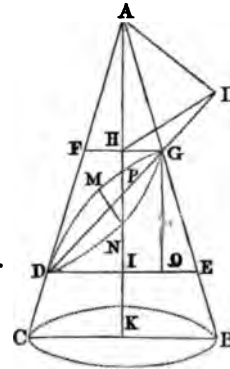
$$\frac{\pi}{12} BC^2 \cdot AK;$$

therefore, by the question,

$$(1) \quad m + n : m :: BC^2 \cdot AK : AL \cdot DG \cdot \sqrt{(DE \cdot FG)}.$$

The triangles DEG and AHL are similar; for, if a circle be described on AG as diameter, the points H and L will be in the circumference, and $AHL = AGL = DGE$, and $HLA = HGA = DEG$; hence

$$(2) \quad DG \cdot AL = DE \cdot AH.$$



Combining equations (1) and (2)

$$(3) \quad m + n : m :: BC^2 . AK : DE . AH \sqrt{(DE . FG)}.$$

Let $DE = 2x$, $FG = 2y$, $BC = 2b$, $AK = ab$; then $AI = ax$, $AH = ay$; and (3) becomes

$$m + n : m :: 4ab^3 : 4axy \sqrt{(xy)}. \quad \therefore xy = b^3 \left(\frac{m}{m+n} \right)^{\frac{1}{2}}.$$

Again, $DP I = DGQ = p$, and $GQ = a(y - x)$ and $DQ = x + y$.

$$\therefore a \tan p (y - x) = x + y; \quad \text{and} \quad x = b \left(\frac{m}{m+n} \right)^{\frac{1}{2}} \sqrt{\left(\frac{a \tan p - 1}{a \tan p + 1} \right)},$$

$$y = b \left(\frac{m}{m+n} \right)^{\frac{1}{2}} \sqrt{\left(\frac{a \tan p + 1}{a \tan p - 1} \right)}.$$

No satisfactory solution for the cases in which the section is a parabola or hyperbola has been received.

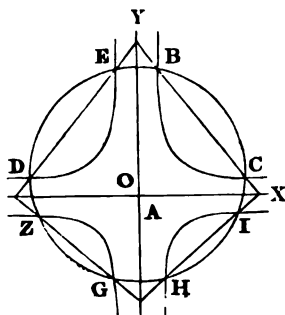
Problem IV. is pronounced to be wrong by several of the competitors.

PRIZE SOLUTION OF PROBLEM V.

By ASHER B. EVANS, Madison University, Hamilton, N. Y.

Describe an ellipse having its foci at any points on one of the asymptotes of an equilateral hyperbola. The ellipse will cut each branch of the hyperbola and its conjugate in two real or imaginary points. Prove that a chord joining one pair of these points will be perpendicular to one of the other similar chords. Prove it first for a circle and then for an ellipse.

Take the asymptotes Ax and Ay as axes; and let $x_1 y_1, x_2 y_2, x_3 y_3,$



$x_4 y_4$ represent B, C, Z, G , the four points common to the hyperbola and the circle; also let θ and θ_1 represent the angles which BC and GZ make with the axis of abscissas; then $\frac{y_2 - y_1}{x_2 - x_1} = \tan \theta$, and $\frac{y_4 - y_3}{x_4 - x_3} = \tan \theta_1$. The equation of the hyperbola is

$$(1) \quad xy = c^2;$$

that of the circle is

$$(2) \quad x^2 + (y - a)^2 = r^2.$$

Combining (1) and (2),

$$(3) \quad y^4 - 2 a y^3 + (a^2 - r^2) y^2 + c^4 = 0.$$

The roots of (3) are y_1, y_2, y_3, y_4 ; hence, by the theory of equations,

$$(4) \quad y_1 y_2 y_3 y_4 = c^4.$$

The co-ordinates of B, C, Z, G must satisfy (1); this condition gives $\tan \theta = -\frac{c^2}{y_1 y_2}$, and $\tan \theta_1 = -\frac{c^2}{y_3 y_4}$.

$$\therefore \tan \theta \tan \theta_1 = \frac{c^4}{y_1 y_2 y_3 y_4} = 1,$$

because (4) exists. Hence $\sin \theta \sin \theta_1 = \cos \theta \cos \theta_1$,

or $\sin \theta \sin \theta_1 - \cos \theta \cos \theta_1 = \cos (\theta + \theta_1) = 0.$

$$\therefore \theta + \theta_1 = 90^\circ.$$

If E, D, I, H be four points common to the conjugate hyperbola and the circle, it is evident from considerations of symmetry that ED and IH , BC and ZG , make equal angles with the axis of x , whenever BC and DE are at right angles respectively with HI and ZG . By substituting the equation of the ellipse for that of the circle in the above proof, it may similarly be shown that this property does *not* hold for the ellipse.

No solution of the problem as amended (see Editorial Items for November number) has been received.

Some of the competitors are requested to take more pains with their penmanship and the arrangement of their solutions.

SIMON NEWCOMB,
W. P. G. BARTLETT,
TRUMAN HENRY SAFFORD.

EXPOSITION OF THE PROCESS OF MATHEMATICAL
DEVELOPMENT.

By JOHN PATERSON, A. M., Albany, N. Y.

[Concluded from Page 61.]

12. THE synchronous and successive series of forces for the time $\psi 1$, are therefore $1 \varphi''$, $1 \psi \varphi'''$, $1 \psi^2 \varphi''$, $1 \psi^3 \varphi'$, $1 \psi^4 \varphi^\circ$, and $1 \varphi''$, $2 \psi \varphi'''$, $3 \psi^2 \varphi''$, $4 \psi^3 \varphi'$, $1 \psi^4 \varphi^\circ$. Applying the successive forces to a succeeding interval $h 1$, of time, and reasoning as above with $\psi 1$, we get the following:—

$\psi 1,$	$h 1,$
$1 \varphi'' = 1 \varphi'' :$	$\frac{1.2}{1} h \varphi''', \quad \frac{1.2.3}{1 \ 2} h^2 \varphi'', \quad \frac{1.2.3.4}{1.2.3} h^3 \varphi', \quad \frac{1.2.3.4.1}{1.2.3.4} h^4 \varphi^\circ = 1 h^4 \varphi^\circ.$
$1 \psi \varphi''' = 2 \psi \varphi''' :$	$\frac{2.3}{1} \psi h \varphi'', \quad \frac{2.3.4}{1.2} \psi h^2 \varphi', \quad \frac{2.3.4.1}{1.2.3} \psi h^3 \varphi^\circ = 4 \psi h^3 \varphi^\circ.$
$1 \psi^2 \varphi'' = 3 \psi^2 \varphi'' :$	$\frac{3.4}{1} \psi^2 h \varphi', \quad \frac{3.4.1}{1.2} \psi^2 h^2 \varphi^\circ = 6 \psi^2 h^2 \varphi^\circ.$
$1 \psi^3 \varphi' = 4 \psi^3 \varphi' :$	$\frac{4.1}{1} \psi^3 h \varphi^\circ = 4 \psi^3 h \varphi^\circ.$
$1 \psi^4 \varphi^\circ = 1 \psi^4 \varphi^\circ.$	

Thus $(\psi^4 + 4 \psi^3 h + 6 \psi^2 h^2 + 4 \psi h^3 + h^4) \varphi^\circ = (\psi + h)^4 \varphi^\circ$ is the final result generated by the force $1 \varphi''$ in the time $(\psi + h) 1$.

13. So far the exposition has treated of free operations only, no restriction whatever being applied to the generating forces, and the result is the normal form of development. The transition to other forms, such as $(\psi \pm h)^{\pm n}$, $(\psi \pm h)^{\pm \frac{p}{q}}$, $a^{\pm \psi}$, $e^{\pm \psi - \frac{1}{2} \psi}$, etc., etc., is effected by the introduction of a suitable condition in each case at the outset of the process.

14. Forces of the first order φ' , originating from different sources, and entering into combination, produce effects or phenomena adapted to observation, the admeasurement of some simple forms of which may be the subject of a future article; and the present one will conclude with an application of the preceding principles to the demonstration of a result hitherto obtained by a different method.

Let $n = 2$: then the force $1\varphi''$ is an independent primitive generator of the first order and second degree, agreeing in character with a *constant force* as usually denominated, such as is the force of gravity at the surface of the earth. This force $1\varphi''$, entering into combination with a reacting force of the same order and degree, which, in the case of the terrestrial force mentioned, exists in the form of a heavy body, for which 1_μ may be written as denoting a unit of mass, will generate force of the second order and first degree φ' , in a first interval of time $\psi'1_t$, in such amount that, with its ultimate value remaining constant, φ' will produce m times the effect in the equal subsequent interval $\psi''1_t$ of time, that was generated by it in the time $\psi'1_t$ of its own genesis : this force of the first degree φ' , in this case, is the velocity (the unit mark of which may be 1_λ) accruing to the body acted upon by the force of gravity ; and the effect of this velocity is the distance described by the falling body, the unit mark of which may be 1_l , and which here takes the place of the force zero φ° .

The constant force $1\varphi''$ generates velocity uniformly, the ultimate value of which is $m1_\lambda$ at the expiration of the time $\psi'1_t$: it is therefore $\frac{m}{2}.1_\lambda$ at the expiration of the time $\frac{1}{2}.\psi'1_t$, with which value it would carry the body 1_μ the distance $\frac{m}{2}\psi^21_l$ in the time $\psi''1_t$, and consequently does carry it the distance $\frac{m}{4}.\psi^21_l$ in the time $\frac{1}{2}.\psi'1_t$, and the m th part of this distance in time $\frac{1}{2}.\psi'1_t$; which again is the distance 1_μ will be carried by the increment of velocity generated by $1\varphi''$ in the second half-interval of time $\frac{1}{2}.\psi'1_t$. This brings the expression $(\frac{1}{4} + \frac{m}{4} + \frac{1}{4})\psi^21_\lambda$; which, when $\psi = 1$, gives the equation $(\frac{1}{4} + \frac{m}{4} + \frac{1}{4})1_\lambda = 1.1_\lambda$, because the effect is to be units in the first unit of time, and the effect is a measure of its cause for that time. The equation gives $m = 2$, and we construct the dynamical triangle : —

$$\frac{\psi \, 1_i}{1 \, \varphi''}; \quad \frac{h \, 1_i}{\frac{1 \cdot 2}{1} h \, 1_i, \frac{1 \cdot 2 \cdot 1}{1 \cdot 2} h^2 \, 1_i} \\ 2 \, \psi \cdot 1_i; \quad \frac{2 \cdot 1}{1} \psi \, h \, 1_i. \\ 1 \, \psi^2 \cdot 1_i.$$

The total is $(1 \, \psi^2 + 2 \, \psi \, h + 1 \, h^2) \, 1_i = (\psi + h)^2 \, 1_i$, the square of the time.

But when the unit of time $1_i = 1$ second, and the unit of distance $1_i = 1$ inch, observation shows that the distance described by any body under the influence of gravity in the first unit of time is approximately 193 inches: then the coefficient of gravity will not be units, but 193, when the unit of time is one second, and the unit of distance is one inch; and the result becomes $(1 \, \psi^2 + 2 \, \psi \, h + h^2)$ 193 inches in the time $(\psi + h) \, 1_i$.

15. The preceding paragraphs have shown the genesis of a series of forces or powers $2 \, \psi \, \varphi^{n-1}$, $3 \, \psi^2 \, \varphi^{n-2}$, $4 \, \psi^3 \, \varphi^{n-3}$, $\dots (n-1) \, \psi^{n-2} \, \varphi''$, $n \, \psi^{n-1} \, \varphi'$, $1 \, \psi^n \, \varphi^0$, by the primitive power $1 \, \varphi^n$, in the time $\psi \, 1_i$. The number n is integral and indefinite, and consequently the series may be prolonged *ad libitum* to the right, or in the positive direction; thus placing $1 \, \psi^n \, \varphi^0$, the zero of force or power, the final phenomenal result, at any required distance below the primitive independent $1 \, \varphi^n$. The numerical coefficients, the exponents of the time ψ , and the indices of the powers φ , all traverse the series by unit steps. Setting aside the *efficient* symbol φ , transition from any term to the next lower one is made by adding unity both to the numerical factor and to the exponent of the literal factor of the *coefficient* of the term selected; or indeed any term whatever is reached at once by adding or subtracting the number appropriate to the interval of transition.

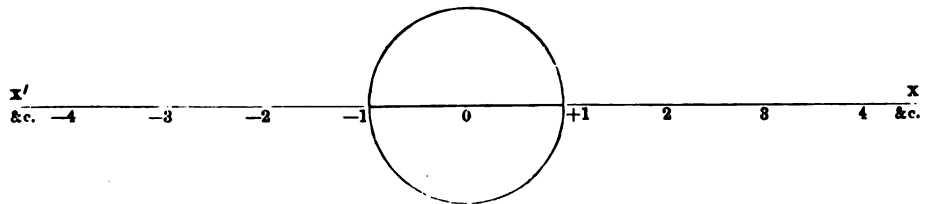
16. Proceeding from right to left in the series (A), and subtracting any convenient number (say 7) from the coefficient and exponent

of the several terms, there arises the series (B), in which terms may be continued in both directions.

- (A) $0\psi^{-1}, 1\psi^0, 2\psi^1, 3\psi^2, 4\psi^3, 5\psi^4, 6\psi^5, 7\psi^6, 8\psi^7, 9\psi^8, 10\psi^9, 11\psi^{10}, 12\psi^{11}, 13\psi^{12}, \&c.$
 (B) $\&c. -6\psi^{-7}, -5\psi^{-6}, -4\psi^{-5}, -3\psi^{-4}, -2\psi^{-3}, -1\psi^{-2}, 0\psi^{-1}, 1\psi^0, 2\psi^1, 3\psi^2, 4\psi^3, 5\psi^4, 6\psi^5, \&c.$
 (C) $\&c. 1\psi^{-7}, 1\psi^{-6}, 1\psi^{-5}, 1\psi^{-4}, 1\psi^{-3}, 1\psi^{-2}, 1\psi^{-1}, 1\psi^0, 1\psi^1, 1\psi^2, 1\psi^3, 1\psi^4, 1\psi^5, \&c.$

The series (B) is merely the series (A) prolonged also in the negative direction, and is therefore the general form of the series expressed in *successive* measures, to the terms of which correspond the *synchronous* measures expressed in the series (C). The coefficient 0 shows that $1\varphi^*$ is not generated, but must be assumed; and indeed $1\varphi^*$ is the unique primitive independent power, or *constant* generating force in mathesis, all beyond zero in (B) proceeding by indefinite regression towards negative infinity. The term $0\psi^{-1}$, which has $1\psi^{-1}\varphi^0$ for the measure of its synchronous value, marks and masks the transition from positive to negative coefficients; for when $\psi = 1$, the coefficient of the term $\frac{1}{2}\psi^0$ expresses the ratio of two equals, while $0.\frac{1}{2}\varphi^*$ indicates equality and opposition in the directions of the two forces which respectively generate the numerator and denominator. This transition from positivity to negativity, and *vice versa*, is effected by passing from linear through angular magnitude, to the opposite direction in space, by means of the revolving radius unity or 1., and comprises the region of logarithms and exponentials, circulars and elliptics, and many definite integrals, etc.

For the time $\psi 1. = 1.1.$, the following diagram exhibits the measures of the coefficient in (B):—



17. That the relations of the terms of (B) are not changed by transition through the critical value, or point of discontinuity, 0, is shown by forming proportions between any two pairs of terms, as thus:—

$$\begin{aligned} 5\psi^4 : 11\psi^{10} :: 3\psi^2 : 9\psi^8, \quad 14\psi^{12} &= 14\psi^{12}; \\ -2\psi^{-3} : 4\psi^3 :: -4\psi^{-5} : 2\psi, \quad 0\psi^{-2} &= 0\psi^{-2}; \\ -2\psi^3 : 4\psi^3 :: 3\psi^2 : 9\psi^8, \quad 7\psi^6 &= 7\psi^6; \end{aligned}$$

where the identical equations are deduced by adding together respectively the coefficients and the exponents of the means and of the extremes. From this it is inferred that under like operations, results deduced respectively from terms on the left and on the right of the zero in (B) are equally true.

18. The power zero φ° may be placed anywhere in the series (B), and then its coefficient must always be replaced by the corresponding term in (C). If φ° be placed to the right of the term $0\psi^{-1}$, the example always has an independent primitive $1\varphi^n$; for instance, the coefficient $3\psi^2$ corresponds to $1\psi^2\varphi^\circ$ as phenomenal result, and to $1\varphi''$ as independent primitive (which is a case furnished by nature in the constant force of gravitation). If $2\psi\varphi^\circ$ were selected, it corresponds to $1\psi\varphi^\circ$; that is, there is no force $2\psi\varphi$ generated, but the result of the genesis is $\psi\varphi^\circ = \psi 1_i$, 1_i being the linear unit, or measure of the effect of a unit 1_λ of constant velocity in the unit of time 1_t , the independent primitive of which is $1\varphi' = 1_\lambda =$ a power of the first order. If $1\psi^0\varphi^\circ$ be selected, then $1\varphi^\circ = 1_i$ is not generated, but is an assumed unit of magnitude.

By these steps, it is seen that the most convenient method of counting will be to place φ° in (C), and, passing diagonally one term to the left in (B), proceed to the primitive $1\varphi^n$. Thus $1\psi^2\varphi^\circ$ recalls the generators $2\psi\varphi'$ and $1\varphi''$; $1\psi\varphi^\circ$ recalls $1\varphi'$; but while

$1\psi^0\varphi^0 = 1\varphi^0$ may be assumed, it has also some kind of power $0\psi^{-1}\varphi$ for its generator; and as 0 is a measure of the ratio of equality, or a logarithm which expresses the number of times that one quantity is greater than another, its interpretation will appear in another place: $0\psi^{-1} = \frac{d\psi}{\psi} = d \cdot \log \psi$.

The term $1\psi^{-1}\varphi^0$ calls up the series

$$-1\psi^{-2}\varphi', -2\psi^{-3}\varphi'', -3\psi^{-4}\varphi''', \dots$$

ad infinitum; and any term further to the left will evoke a similar series of negative coefficients and exponents, with coefficients ascending indefinitely upwards from units in order, always commencing in (B) with φ' one step to the left of the term φ^0 selected in (C).

19. The scheme for the development of the function $(\psi + h)^n$ is as follows:—

$\psi 1_n :$	$h 1_n.$
$1\varphi^n; \frac{1.2}{1}h\varphi^{n-1}, \frac{1.2.3}{1.2}h^2\varphi^{n-2}, \frac{1.2.3.4}{1.2.3}h^3\varphi^{n-3}, \frac{1.2.3.4.5}{1.2.3.4}h^4\varphi^{n-4}, \&c.$	
$2\psi\varphi^{n-1}; \frac{2.3}{1}\psi h\varphi^{n-2}, \frac{2.3.4}{1.2}\psi h^2\varphi^{n-3}, \frac{2.3.4.5}{1.2.3}\psi h^3\varphi^{n-4}, \&c.$	
$3\psi^2\varphi^{n-2}; \frac{3.4}{1}\psi^2 h\varphi^{n-3}, \frac{3.4.5}{1.2}\psi^2 h^2\varphi^{n-4}, \&c.$	
$4\psi^3\varphi^{n-3}; \frac{4.5}{1}\psi^3 h\varphi^{n-4}, \&c.$	
$5\psi^4\varphi^{n-4}; \&c.$	
$\&c.$	

On reaching the power $\varphi^{n-n} = \varphi^0$, all the terms of the lower diagonal series (the hypotenuse of the arithmetical triangle) are to be divided by the last factor of the numerators to get the final result.

Let $n = 4$: the triangle becomes

$\psi 1_i.$	$h 1_i.$				
$1 \varphi^{iv}; \frac{1.2}{1} h \varphi''', \frac{1.2.3}{1.2} h^2 \varphi'', \frac{1.2.3.4}{1.2.3} h^3 \varphi', \frac{1.2.3.4.1}{1.2.3.4} h^4 \varphi^{\circ} = 1 h^4 \varphi^{\circ}.$					
$2 \psi \varphi'''; \frac{2.3}{1} \psi h \varphi'', \frac{2.3.4}{1.2} \psi h^2 \varphi', \frac{2.3.4.1}{1.2.3} \psi h^3 \varphi^{\circ} = 4 \psi h^3 \varphi^{\circ}.$					
$3 \psi^2 \varphi''; \frac{3.4}{1} \psi^2 h \varphi', \frac{3.4.1}{1.2} \psi^2 h^2 \varphi^{\circ} = 6 \psi^2 h^2 \varphi^{\circ}.$					
$4 \psi^3 \varphi'; \frac{4.1}{1} \psi^3 h \varphi^{\circ} = 4 \psi^3 h \varphi^{\circ}.$					
$1 \psi^4 \varphi^{\circ}.$					

Which is $(\psi^4 + 4 \psi^3 h + 6 \psi^2 h^2 + 4 \psi h^3 + h^4) \varphi^{\circ} = (\psi + h)^4 \varphi^{\circ}.$

During the past time $\psi 1_i$, the independent primitive $1 \varphi^{iv}$ of the fourth order has generated the final result $1 \psi^4 \varphi^{\circ}$, or has caused the unit of mass 1_m to describe the distance $1 \psi^4 1_i$ (or has generated the distance $1 \psi^4 1_i$); and has also generated the forces or powers $4 \psi^3 \varphi'$, $3 \psi^2 \varphi''$, and $2 \psi \varphi'''$, respectively of the first, second, and third orders, which become constant generators at the expiration of the time $\psi 1_i$. The power $4 \psi^3 \varphi'$ is a constant velocity $4 \psi^3 1_i$, which will of itself generate the distance $4 \psi^3 h 1_i$ in the future time $h 1_i$: it is the only velocity in existence in the system at the commencement of the time $h 1_i$, and is the differential coefficient of ψ^4 , which gives the motion in the tangent of the angle that the curve makes with the axis of abscissa at the point ψ , rectangular co-ordinates understood. The power $3 \psi^2 \varphi''$ is a constant force which generates velocity uniformly; and as it has itself generated the velocity $4 \psi^3 1_i$ during its growth, it now adds the distance $6 \psi^2 h^2 1_i$ to the foregoing terms, by means of the velocity $12 \psi^2 h 1_i$ which it generates in the time $h 1_i$. The power $2 \psi \varphi'''$ is a constant force of the third order, which generates force of the second order uniformly, which generates velocity increasingly during the time $h 1_i$, and adds the distance $4 \psi h^3 1_i$ to the foregoing terms. And lastly the primitive power $1 \varphi^{iv}$ itself generates the distance $h^4 1_i$ in the time $h 1_i$, just as the distance $\psi^4 1_i$ was its final result in the time $\psi 1_i$.

20. Positive development begins at unity, proceeds towards positive infinity, and can be arrested at any term. Conversely, negative development begins at negative infinity, but may be taken up at any term, and concludes at unity. A closing portion of the scheme follows : —

ψ 1.,	h 1.,			
$\&c.$	$\&c.$	$\&c.$	$\&c.$	$\&c.$
$-7 \psi^{-8} \varphi^{vii};$	$\frac{-7!}{-5!} \psi^{-8} h \varphi^{vi},$	$\frac{-7!}{-4! 2!} \psi^{-8} h^2 \varphi^v, \dots$	$\frac{-7! 1}{7!} \psi^{-8} h^7 \varphi^o.$	
$-6 \psi^{-7} \varphi^{vi};$	$\frac{-6!}{-4!} \psi^{-7} h \varphi^v,$	$\frac{-6!}{-3! 2!} \psi^{-7} h^2 \varphi^{iv}, \dots$	$\frac{-6! 1}{6!} \psi^{-7} h^6 \varphi^o.$	
$-5 \psi^{-6} \varphi^v;$	$\frac{-5!}{-3!} \psi^{-6} h \varphi^{iv},$	$\frac{-5!}{-2! 2!} \psi^{-6} h^2 \varphi^{iii}, \dots$	$\frac{-5! 1}{5!} \psi^{-6} h^5 \varphi^o.$	
$-4 \psi^{-5} \varphi^{iv};$	$\frac{-4!}{-2!} \psi^{-5} h \varphi^{iii},$	$\frac{-4!}{-1.2!} \psi^{-5} h^2 \varphi^{ii},$	$\frac{-4!}{3!} \psi^{-5} h^3 \varphi^i,$	$\frac{-4! 1}{4!} \psi^{-5} h^4 \varphi^o.$
$-3 \psi^{-4} \varphi^{iii};$	$\frac{(-3)(-2)}{1} \psi^{-4} h \varphi^{ii},$	$\frac{-3!}{1.2} \psi^{-4} h^2 \varphi^i,$	$\frac{-3! 1}{3!} \psi^{-4} h^3 \varphi^o.$	
$-2 \psi^{-3} \varphi^{ii};$	$\frac{(-2)(-1)}{1} \psi^{-3} h \varphi^i,$	$\frac{(-2)(-1)}{1} \psi^{-3} h^2 \varphi^o.$		
$-1 \psi^{-2} \varphi^i;$	$\frac{(-1) 1}{1} \psi^{-2} h \varphi^o.$			
$+1 \psi^{-1} \varphi^o;$				

in which the sign $n!$, or $\pm \frac{2n+1}{2}!$, as usual denotes the continued product $1.2\dots n$; or $\frac{1}{2} \cdot \frac{3}{2} \dots \frac{2n+1}{2}$; or $(-\frac{1}{2})(-\frac{3}{2})\dots(-\frac{2n+1}{2})$

The first diagonal line

$$\left(\frac{1}{\psi} - \frac{1}{\psi^2} h + \frac{1}{\psi^3} h^2 - \frac{1}{\psi^4} h^3 + \frac{1}{\psi^5} h^4 - \frac{1}{\psi^6} h^5 + \frac{1}{\psi^7} h^6 - \frac{1}{\psi^8} h^7 \right) \varphi^o$$

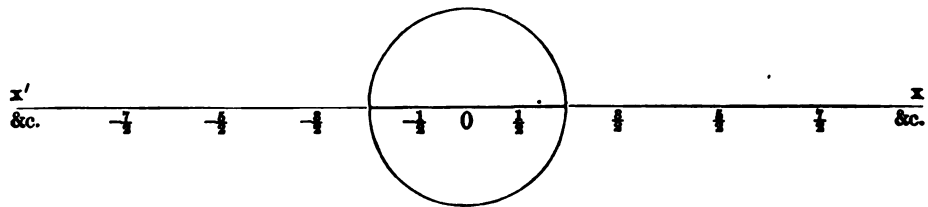
is the development of $(\psi + h)^{-1}$ carried through eight terms. But here, as in the case of positive development, the power zero φ^o may be placed on any term of the first perpendicular line; where the

corresponding diagonal line, being divided throughout by the numerical factor common to every term, will express the development of $(\psi + h)$ to the index of ψ in the first term. For example, place φ° upon the fourth term from the last, and there appears :—

$\psi \ 1_{\iota}.$	$h \ 1_{\iota}.$				
&c.	&c.	&c.	&c.	&c.	
$-7 \psi^{-8} \varphi^{iv};$	$\frac{-7!}{5!} \psi^{-8} h \varphi''';$	$\frac{-7!}{4! \ 2!} \psi^{-8} h^2 \varphi'';$	$\frac{-7!}{3! \ 3!} \psi^{-8} h^3 \varphi';$	$\frac{-7! \ 1}{3! \ 4!} \psi^{-8} h^4 \varphi^\circ.$	
$-6 \psi^{-7} \varphi''';$	$\frac{-6!}{4!} \psi^{-7} h \varphi'';$	$\frac{-6!}{3! \ 2!} \psi^{-7} h^2 \varphi';$	$\frac{-6! \ 1}{3! \ 3!} \psi^{-7} h^3 \varphi^\circ.$		
$-5 \psi^{-6} \varphi'';$	$\frac{(-5)(-4)}{1} \psi^{-6} h \varphi';$	$\frac{(-5)(-4) \ 1}{1 \cdot 2} \psi^{-6} h^2 \varphi^\circ.$			
$-4 \psi^{-5} \varphi';$	$\frac{(-4) \ 1}{1} \psi^{-5} h \varphi^\circ.$				
$+1 \psi^{-4} \varphi^\circ.$					

Total $\left(\frac{1}{\psi^4} - 4 \frac{h}{\psi^5} + 10 \frac{h^2}{\psi^6} - 20 \frac{h^3}{\psi^7} + 35 \frac{h^4}{\psi^8}\right) \varphi^\circ = (\psi + h)^{-4} \varphi^\circ$ to five terms.

21. To pass from integral to fractional development, observe, that retaining the unit interval 1, between the several powers at the expiration of the first unit of time 1, they may be placed at fractional distances from the origin O or 0, as in Fig. 2. Subtract any



fractional number $\frac{p}{q}$, say $\frac{1}{2}$, from the coefficients and exponents of the terms of the successive integral series (B), and there arises the successive fractional series (D), to which corresponds the synchronous series (E):—

$$(B) \quad \&c. -4\psi^{-5}, -3\psi^{-4}, -2\psi^{-3}, -1\psi^{-2}, 0\psi^{-1}, 1\psi^0, 2\psi^1, 3\psi^2, 4\psi^3, 5\psi^4, 6\psi^5, \quad \&c.$$

$$(D) \quad \&c. -\frac{1}{2}\psi^{-\frac{1}{2}}, -\frac{1}{2}\psi^{-\frac{3}{2}}, -\frac{3}{2}\psi^{-\frac{5}{2}}, -\frac{7}{2}\psi^{-\frac{7}{2}}, -\frac{9}{2}\psi^{-\frac{9}{2}}, -\frac{3}{2}\psi^{-\frac{11}{2}}, -\frac{1}{2}\psi^{-\frac{13}{2}}, \frac{1}{2}\psi^{-\frac{15}{2}}, \frac{3}{2}\psi^{-\frac{17}{2}}, \frac{7}{2}\psi^{-\frac{19}{2}}, \frac{9}{2}\psi^{-\frac{21}{2}}, \&c.$$

$$(E) \quad \&c. 1\psi^{-\frac{1}{2}}, 1\psi^{-\frac{3}{2}}, 1\psi^{-\frac{5}{2}}, 1\psi^{-\frac{7}{2}}, 1\psi^{-\frac{9}{2}}, 1\psi^{-\frac{11}{2}}, 1\psi^{-\frac{13}{2}}, 1\psi^{-\frac{15}{2}}, 1\psi^{-\frac{17}{2}}, 1\psi^{-\frac{19}{2}}, 1\psi^{-\frac{21}{2}}, \&c.$$

1°. Place the zero of power at the term $1\psi^{\frac{1}{2}}$, and get the triangle :

$\psi \ 1_{\frac{1}{2}}$	$h \ 1_{\frac{1}{2}}$			
&c.	&c.	&c.	&c.	
$-\frac{5}{2}\psi^{-\frac{1}{2}}\varphi^{IV};$	$-\frac{5}{2}\psi^{-\frac{1}{2}}h\varphi''';$	$-\frac{5}{2}\psi^{-\frac{1}{2}}h^2\varphi'';$	$-\frac{5}{2}\psi^{-\frac{1}{2}}h^3\varphi';$	$-\frac{5}{2}\psi^{-\frac{1}{2}}h^4\varphi^0 = -\frac{1 \cdot 1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} (\frac{1}{2})^4 \frac{h^4}{\psi^{\frac{1}{2}}} \varphi^0.$
$-\frac{3}{2}\psi^{-\frac{3}{2}}\varphi''';$	$-\frac{3}{2}\psi^{-\frac{3}{2}}h\varphi'';$	$-\frac{3}{2}\psi^{-\frac{3}{2}}h^2\varphi';$	$-\frac{3}{2}\psi^{-\frac{3}{2}}h^3\varphi^0 = +\frac{1 \cdot 1 \cdot 3}{3!} (\frac{1}{2})^3 \frac{h^3}{\psi^{\frac{3}{2}}} \varphi^0.$	
$-\frac{1}{2}\psi^{-\frac{5}{2}}\varphi'';$	$-\frac{1}{2}\psi^{-\frac{5}{2}}h\varphi';$	$-\frac{1}{2}\psi^{-\frac{5}{2}}h^2\varphi^0 = -\frac{1 \cdot 1}{2!} (\frac{1}{2})^2 \frac{h^2}{\psi^{\frac{5}{2}}} \varphi^0.$		
$+\frac{1}{2}\psi^{-\frac{7}{2}}\varphi';$	$+\frac{1}{2}\psi^{-\frac{7}{2}}h\varphi^0 = +\frac{1}{2} (\frac{1}{2}) \frac{h}{\psi^{\frac{7}{2}}} \varphi^0.$			

$$1\psi^{\frac{1}{2}}\varphi^0.$$

Which is the development of $+\sqrt{(\psi+h)}$ to five terms.

2°. Place the power zero at the term $1\psi^{-\frac{1}{2}}$, and get

$\psi \ 1_{\frac{1}{2}}$	$1 \ h_{\frac{1}{2}}$			
&c.	&c.	&c.	&c.	
$-\frac{7}{2}\psi^{-\frac{1}{2}}\varphi^{IV};$	$-\frac{7}{2}\psi^{-\frac{1}{2}}h\varphi''';$	$-\frac{7}{2}\psi^{-\frac{1}{2}}h^2\varphi'';$	$-\frac{7}{2}\psi^{-\frac{1}{2}}h^3\varphi';$	$-\frac{7}{2}\psi^{-\frac{1}{2}}h^4\varphi^0 = +\frac{1 \cdot 3 \cdot 5 \cdot 7}{4!} (\frac{1}{2})^4 \frac{h^4}{\psi^{\frac{1}{2}}} \varphi^0.$
$-\frac{5}{2}\psi^{-\frac{3}{2}}\varphi''';$	$-\frac{5}{2}\psi^{-\frac{3}{2}}h\varphi'';$	$-\frac{5}{2}\psi^{-\frac{3}{2}}h^2\varphi';$	$-\frac{5}{2}\psi^{-\frac{3}{2}}h^3\varphi^0 = -\frac{1 \cdot 3 \cdot 5}{3!} (\frac{1}{2})^3 \frac{h^3}{\psi^{\frac{3}{2}}} \varphi^0.$	
$-\frac{3}{2}\psi^{-\frac{5}{2}}\varphi'';$	$-\frac{3}{2}\psi^{-\frac{5}{2}}h\varphi';$	$-\frac{3}{2}\psi^{-\frac{5}{2}}h^2\varphi^0 = +\frac{1 \cdot 3}{2!} (\frac{1}{2})^2 \frac{h^2}{\psi^{\frac{5}{2}}} \varphi^0.$		
$-\frac{1}{2}\psi^{-\frac{7}{2}}\varphi';$	$-\frac{1}{2}\psi^{-\frac{7}{2}}h\varphi^0 = -\frac{1}{2} (\frac{1}{2}) \frac{h}{\psi^{\frac{7}{2}}} \varphi^0.$			

$$1\psi^{-\frac{1}{2}}\varphi^0.$$

Which is the development of $-\sqrt{(\psi+h)}$ to five terms.

The development of $(\psi+h)^{\frac{1}{2}}$ is found as follows :—

$\psi 1,.$	$h 1,.$
etc.	etc.
$-\frac{7}{2}\psi^{-\frac{3}{2}}\varphi^{\text{viii}};$	$-\frac{7}{2}\psi^{-\frac{3}{2}}h\varphi^{\text{viii}}, \dots -\frac{7!}{8!}\psi^{-\frac{3}{2}}h^8\varphi^{\circ} = +\frac{7.5.3.1.1.3.5.7}{8!}\left(\frac{1}{2}\right)^8\frac{h^8}{\psi^{\frac{3}{2}}}\varphi^{\circ}.$
$-\frac{5}{2}\psi^{-\frac{5}{2}}\varphi^{\text{vi}};$	$-\frac{5}{2}\psi^{-\frac{5}{2}}h\varphi^{\text{vi}}, \dots -\frac{5!}{7!}\psi^{-\frac{5}{2}}h^7\varphi^{\circ} = -\frac{7.5.3.1.1.3.5}{7!}\left(\frac{1}{2}\right)^7\frac{h^7}{\psi^{\frac{5}{2}}}\varphi^{\circ}.$
$-\frac{3}{2}\psi^{-\frac{7}{2}}\varphi^{\text{iv}};$	$-\frac{3}{2}\psi^{-\frac{7}{2}}h\varphi^{\text{iv}}, \dots -\frac{3!}{6!}\psi^{-\frac{7}{2}}h^6\varphi^{\circ} = +\frac{7.5.3.1.1.3}{6!}\left(\frac{1}{2}\right)^6\frac{h^6}{\psi^{\frac{7}{2}}}\varphi^{\circ}.$
$-\frac{1}{2}\psi^{-\frac{9}{2}}\varphi^{\text{ii}};$	$-\frac{1}{2}\psi^{-\frac{9}{2}}h\varphi^{\text{ii}}, \dots -\frac{1!}{5!}\psi^{-\frac{9}{2}}h^5\varphi^{\circ} = -\frac{7.5.3.1.1}{5!}\left(\frac{1}{2}\right)^5\frac{h^5}{\psi^{\frac{9}{2}}}\varphi^{\circ}.$
$+\frac{1}{2}\psi^{-\frac{11}{2}}\varphi^{\text{iv}};$	$\frac{1}{2}\psi^{-\frac{11}{2}}h\varphi^{\text{iv}}, \dots \frac{1!}{4!}\psi^{-\frac{11}{2}}h^4\varphi^{\circ} = +\frac{7.5.3.1}{4!}\left(\frac{1}{2}\right)^4\frac{h^4}{\psi^{\frac{11}{2}}}\varphi^{\circ}.$
$+\frac{3}{2}\psi^{-\frac{13}{2}}\varphi^{\text{vi}};$	$\frac{3}{2}\psi^{-\frac{13}{2}}h\varphi^{\text{vi}}, \dots \frac{3!}{2!}\psi^{-\frac{13}{2}}h^2\varphi^{\circ} = +\frac{7.5.3}{2!}\left(\frac{1}{2}\right)^3\psi^{-\frac{13}{2}}h^2\varphi^{\circ}.$
$+\frac{5}{2}\psi^{-\frac{15}{2}}\varphi^{\text{viii}};$	$\frac{5}{2}\psi^{-\frac{15}{2}}h\varphi^{\text{viii}}, \dots \frac{5!}{2!}\psi^{-\frac{15}{2}}h^2\varphi^{\circ} = +\frac{7.5}{1.2}\left(\frac{1}{2}\right)^2\psi^{-\frac{15}{2}}h^2\varphi^{\circ}.$
$+\frac{7}{2}\psi^{-\frac{17}{2}}\varphi^{\text{x}};$	$\frac{7}{2}\psi^{-\frac{17}{2}}h\varphi^{\circ} = +\frac{7}{1}\left(\frac{1}{2}\right)\psi^{-\frac{17}{2}}h\varphi^{\circ}.$
$1\varphi^{\frac{1}{2}}\varphi^{\circ}.$	

Thus positive fractional development is not arrested, but passes over zero by revolution.

22. When h is negative, the future time $h 1,$, after the expiration of the time $\psi 1,$, is changed for past time $-h 1,$, the results of which are to be subtracted from those of the full time $\psi 1,$. The factors of the numerators, even when under the negative sign, always tend in the positive direction; those of the denominators also advance for time *future* to $\psi 1,$; consequently they will reverse their direction for time that is *past* when $\psi 1,$ has expired. Then for the development of $(\psi - h)^{\frac{1}{2}}$, the form will be

$\psi 1,.$	$h 1,.$
$1\varphi^{\text{iv}};$	$\frac{2!}{-1}h\varphi^{\text{iv}}, \frac{3!}{-2!}h^2\varphi^{\text{iv}}, \frac{4!}{-3!}h^3\varphi^{\text{iv}}, \frac{4!1}{-4!}h^4\varphi^{\circ} = +1h^4\varphi^{\circ}.$
$2\psi^2\varphi^{\text{vi}};$	$\frac{3!}{-1}\psi h\varphi^{\text{vi}}, \frac{4!}{-2!}\psi h^2\varphi^{\text{vi}}, \frac{4!1}{-3!}\psi h^3\varphi^{\circ} = -4\psi h^3\varphi^{\circ}.$
$3\psi^2\varphi^{\text{iv}};$	$\frac{4!}{-1.2!}\psi^2h\varphi^{\text{iv}}, \frac{4!1}{-2!2!}\psi^2h^2\varphi^{\circ} = +6\psi^2h^2\varphi^{\circ}.$
$4\psi^2\varphi^{\text{ii}};$	$\frac{4.1}{-1}\psi^2h\varphi^{\circ} = -4\psi^2h\varphi^{\circ}.$
$1\psi^4\varphi^{\circ}.$	

And for $(\psi - h)^{-4}$, the form will be

$\psi 1_1.$	$h 1_1.$			
&c.	&c.	&c.	&c.	
$-7 \psi^{-8} \varphi'''';$	$\frac{-7!}{-5!-1} \psi^{-8} h \varphi'''';$	$\frac{-7!}{-4!-2!} \psi^{-8} h^2 \varphi'''';$	$\dots$	$\frac{-7!1}{-3!-4!} \psi^{-8} h^4 \varphi^\circ = + \frac{35 h^4}{\psi^8} \varphi^\circ.$
$-6 \psi^{-7} \varphi''''';$	$\frac{-6!}{-3!-1} \psi^{-7} h \varphi''''';$	$\frac{-6!}{-3!-2!} \psi^{-7} h^2 \varphi''''';$	$\frac{-6!1}{-2!-3!} \psi^{-7} h^3 \varphi^\circ = + \frac{20 h^3}{\psi^7} \varphi^\circ.$	
$-5 \psi^{-6} \varphi'''';$	$\frac{-5!}{-3!-1} \psi^{-6} h \varphi'''';$	$\frac{-5!1}{-3!-2!} \psi^{-6} h^2 \varphi^\circ = + \frac{10 h^2}{\psi^6} \varphi^\circ.$		
$-4 \psi^{-5} \varphi''''';$	$\frac{-4!1}{-1} \psi^{-5} h \varphi^\circ = + \frac{4 h}{\psi^5} \varphi^\circ.$			
$+1 \psi^{-4} \varphi^\circ$				

When the exponent of ψ is negative, uniform semicircular rotation commences with the beginning of the time $\psi 1_1$; when the coefficient of h is negative, an equal rotation begins with the time $h 1_1$.

NOTES ON PROBABILITIES.

By SIMON NEWCOMB, Cambridge, Mass.

THE problem with which the article on probabilities, in the Mathematical Monthly for May, 1860, was closed, belongs to a class which has been much discussed. We shall here repeat it, and then show a variation in it which leads to a result, which, at first sight, seems paradoxical.

An urn contains 100 balls, 1 of which is black, and 99 white; an individual, who falsifies one time in ten, draws a ball, and asserts that it is black. What is the probability that he tells the truth? We have found that it is $\frac{1}{12}$.

But suppose that the balls are numbered from 1 to 100, and that he asserts that the ball No. 79 is drawn. What is the probability that he tells the truth?

The solution is exhibited in tabular form, as follows. The first column contains the separate hypotheses which it is possible to make, column p gives the *a priori* probabilities of these hypotheses, and column q the probability of the observed event on the several hypotheses, as explained in the number of the Mathematical Monthly above referred to.

c	p	q	$p q$
No. 79 drawn	$\frac{1}{100}$	$\frac{9}{10}$	$\frac{9}{1000}$
Some other No. drawn,	$\frac{99}{100}$	$\frac{1}{990}$	$\frac{1}{1000}$

Whence the probability that No. 79 was drawn, or that the truth is told, is, by § 24, $\frac{\frac{9}{1000}}{\frac{9}{1000} + \frac{1}{1000}} = \frac{9}{10}$.

The value of q , on the second hypothesis, is obtained as follows:— Suppose that some other number than 79 is drawn. Then, if he is going to falsify, he has the 99 numbers which were not drawn, to choose from. Since there is supposed to be no reason why he should say 79 rather than any number, the probability that he will select this number from the 99 not drawn is $\frac{1}{99}$. But the probability that he will falsify in the first place is $\frac{1}{10}$. Whence the probability of saying 79 falsely is $\frac{1}{990}$.

Now the apparent paradox is, that the probability that he has told the truth is so vastly different when he says that a single black ball has been drawn from 100, from what it is when he says that the single number 79 has been so drawn, being $\frac{1}{10}$ in the first case, and $\frac{9}{10}$ in the second. Several forms of explanation may be given, and the reader can select that which seems to him most clear.

1. The difference of the results proceeds from the circumstance that he is supposed to have but two colors to choose from if he falsifies, viz. white and black, while he has 100 numbers. Since a white ball is drawn 99 times out of a hundred, and since he falsifies about one time in ten, he will have said black *about* 11 times in

every hundred, ten times falsely, and once truly. Whence I conclude that when he says black he does not tell the truth, for if I believed him to tell the truth every time, I should have to believe that an excessively improbable number of black balls had been drawn; instead of which I knew beforehand that he would make some ten or twelve false statements respecting the drawing of the black ball. But if 100 numbered balls have been drawn, his ten falsifications are supposed to have been divided amongst the 100 numbers I know not how, whence the chance that any one number was called falsely is only $\frac{1}{10}$.

If, however, I found that some one number, such as 79, had been called 10 or 12 times, I should conclude that the witness was disposed to falsify by saying 79, and thus I would not believe him, and the second case would be reduced to the first. Again, if there were a hundred colors from which he might choose, I would believe him when he said black, because I should conclude that if he were going to falsify, he would probably select some of the other 99 colors, and thus the first case of the problem would become identical with the second.

The following is an interesting problem, and might be useful. A bag contains an unknown number of tickets, numbered from one upwards. A person draws a ticket, notes the number, and replaces it; then repeats the operation till he has drawn s times. What judgment can he form as to the number of tickets in the bag?

Let p_n represent the *a priori* probability that the bag contains just n tickets, and no more. Let i be the largest number drawn in the s drawings. The number of tickets, then, cannot be less than i . We need not know any of the drawn numbers except the largest, because after we know this, every combination of smaller numbers will be equally probable on every admissible hypothesis, and will therefore be of no assistance in judging of these hypotheses. We

may then be supposed to know that s tickets have been drawn, and that the largest number observed is i . Now let us make our hypotheses. Suppose that the bag contains just i tickets. The probability that No. i would *not* be drawn on any one drawing is $\frac{i-1}{i}$. Hence the probability that it would not be drawn at all is $\frac{(i-1)^s}{i^s}$; hence the probability that it would be drawn on one or more trials is $\frac{i-(i-1)^s}{i^s}$. Again, if we suppose the bag to contain $i+h$ tickets, the probability that i would be the largest number drawn would be equal to the product of the probability that none of the h numbers above i would be drawn into the probability that i would be drawn if none of the h were drawn. The first of these probabilities is $\frac{i^s}{(i+h)^s}$; whence the probability that i would be the largest number is $\frac{i-(i-1)^s}{(i+h)^s}$. We can now arrange the quantities as before. We put for brevity $i^s - (i-1)^s = \Delta$.

c	p	q	$p q$
i balls	p_i	$\frac{\Delta}{i^s}$	$\frac{\Delta p_i}{i^s}$
$(i+1)$ balls	p_{i+1}	$\frac{\Delta}{(i+1)^s}$	$\frac{\Delta p_{i+1}}{(i+1)^s}$
$(i+2)$ balls	p_{i+2}	$\frac{\Delta}{(i+2)^s}$	$\frac{\Delta p_{i+2}}{(i+2)^s}$
&c.	&c.	&c.	&c.

Put P_n for the final probability that the bag contains just n tickets, and we have, by the general formula,

$$(1) \quad P_{i+h} = \frac{\Delta p_{i+h}}{(i+h)^s \left\{ \frac{\Delta p_i}{i^s} + \frac{\Delta p_{i+1}}{(i+1)^s} + \&c. \right\}} = \frac{p_{i+h}}{(i+h)^s \cdot \Sigma}.$$

putting, for brevity,

$$\Sigma = \frac{p_i}{i^s} + \frac{p_{i+1}}{(i+1)^s} + \frac{p_{i+2}}{(i+2)^s} + \dots$$

By taking the sum of a sufficient number of the values of P_{i+h} from $h=0$ upwards to make $\frac{1}{2}$, we should find that number which

it was an even chance whether the balls would fall short of, or exceed. In getting the value of Σ , the series extends to all the numbers which it is possible that the bag should contain.

As a particular case, suppose that all numbers up to a certain limit are, before the drawing, equally probable. The values of p_n will then be all equal; and by putting

$$S = \frac{1}{i^s} + \frac{1}{(i+1)^s} + \frac{1}{(i+2)^s} + \&c.,$$

for all possible values, we shall have

$$(2) \quad P_i = \frac{1}{S i^s}, \quad P_{i+1} = \frac{1}{S (i+1)^s}, \dots P_{i+h} = \frac{1}{S (i+h)^s}.$$

We see from these expressions that the highest number drawn is the most probable of *any single number*, so that if we were required to guess the exact number contained in the bag, we ought to guess this one. But if i and h are not very small, several numbers above i will each be almost as probable as i itself, and the guess would therefore probably be wrong. But it would be more satisfactory to find the probability that the number is contained between given limits. Thus, the probability that it does not exceed $i+k$ is, by summing the separate probabilities,

$$(3) \quad \frac{1}{S} \left\{ \frac{1}{i^s} + \frac{1}{(i+1)^s} + \frac{1}{(i+2)^s} + \dots + \frac{1}{(i+k)^s} \right\}.$$

As a numerical example, suppose that all numbers are equally before drawing, and that upon drawing four times, the highest number drawn is 10. Then we have $i=10$, $s=4$, $h=\infty$.

$$\text{Also } S = \sum_1^\infty \frac{1}{x^4} = \sum_1^{10} \frac{1}{x^4} + \frac{\pi^4}{90} - \sum_1^{10} \frac{1}{x^4} = 0.000386650,$$

$$\frac{1}{i^s} = 0.0001 = \frac{1}{10^4}, \quad \frac{1}{(i+1)^s} = 0.0000683, \quad \frac{1}{(i+2)^s} = 0.0000482.$$

The sum of the first two numbers is rather more than $\frac{1}{4}S$, and

the sum of the first three rather greater. It is therefore rather less than an even chance that the bag contains 11 tickets, or less, but rather more than an even chance that it does not contain more than 12. If we make $k = 10$ in (3), the resulting value of the probability is $\frac{3481}{666}$. We might therefore wager more than 8 against 1 that the bag did not contain more than 20 balls.

As another numerical example, suppose that I read in a paper the addresses of ten persons, all at the same post-office, but having different boxes, of which the numbers are given. The highest number was 200. What is the probable number of boxes which the office contains, supposing that one box is as likely to be let to any person as another?

To solve this, we observe that the sum of the series

$$1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \dots + \frac{1}{(x-1)^{10}},$$

is $\frac{\pi^{10}}{93555} - \frac{1}{9x^9} - \frac{1}{2x^{10}} - \frac{10}{12x^{11}} + \frac{132}{72x^{12}} - \&c.$

The sum from x to infinity is, therefore,

$$\frac{1}{x^{10}} + \frac{1}{(x+1)^{10}} + \frac{1}{(x+2)^{10}} + \dots = \frac{1}{9x^9} + \frac{1}{2x^{10}} + \frac{10}{12x^{11}} - \frac{11}{6x^{12}} + \&c.$$

and by taking the difference of these values for different values of x , we can find the sum of the series between finite limits. Representing by $\Sigma_{i, i'}$ the sum of the series between the limits i and i' inclusive, we have

$$\Sigma_{200, \infty} = 22194,$$

$$\Sigma_{200, 215} = 11110,$$

in units of the twenty-sixth place of decimals. There is, therefore, about an even chance that the post-office does not contain more than 215 boxes.

When i is so large that we may neglect all terms after the first in the expressions for the sum of the series, we shall have

$$S = \Sigma_{i=\infty} = \frac{1}{(S-1)i^{s-1}} \quad \Sigma_{i,i'} = \frac{1}{(S-1)i^{s-1}} - \frac{1}{(S-1)(i'-1)^{s-1}};$$

which, from equation (3), gives for the probability that the number required is contained between the limits i and i' inclusive,

$$P_{i,i'} = 1 - \frac{i^{s-1}}{(i'+1)^{s-1}}.$$

If we wish to find the limits within which there is a given probability that the number is contained, we obtain from the last equation,

$$i' = \frac{i}{(1 - P_{i,i})^{s-1}} - 1.$$

Mathematical Monthly Notices.

A Treatise on Astronomy, Descriptive, Theoretical, and Physical, Designed for Schools, Academies, and Private Students. By H. N. ROBINSON, A. M., Formerly Professor of Mathematics in the United States Navy; Author of a Treatise on Arithmetic, Algebra, Geometry, Trigonometry, Surveying, Calculus, Natural Philosophy, &c., &c. New York: Ivison & Phinney. 1858.

This treatise professes to give an exposition, not only of the operations of practical astronomy, but of the motions of the heavenly bodies as deduced from the law of gravitation.

A few extracts from the work would be amply sufficient to enable the professional astronomer to judge of the author's success in carrying out his plan, but as teachers and students generally will wish to form such a judgment, we shall add such comments as the extracts suggest.

On page 67 it is stated that, "The shape of the earth, thus ascertained by actual measurement, is just what theory would give to a body of water equal to our globe, and revolving on an axis in 24 hours." Unfortunately, the shape of our earth is very far indeed from being that which theory would give to a body of water "equal to our globe, and revolving on an axis in 24 hours," as any one will find by putting into numbers the results given by any writer on the figure of a homogeneous revolving fluid mass in equilibrium; e.g. DE PONTÉCOULANT, *Système du Monde*, Vol. II.

On page 79 and 80 the method given for computing the relation between the horizontal parallax and semidiameter of the moon is entirely erroneous in theory, from the *tangents* of those quantities being used instead of the *sines*. Practically, the mistake is of slight moment.

On page 89 we find the statement that the real velocity of the earth in the orbit "varies in the inverse ratio as the radius vector; or it varies directly as the apparent diameter of the sun." Here the author has forgotten that the earth does not always move in a direction at right an-

gles to that of the line drawn through the earth and the sun; and his statement is not true, unless the velocity is first multiplied by the cosine of this angle.

The following explanation will strike the physicist as being both unique and original:—

"The inquiry is sometimes made why we do not have the warmest weather about the summer solstice, and the coldest weather about the time of the winter solstice.

"This would be the case if the sun immediately ceased to give extra warmth, on arriving at the summer solstice; but if it could radiate extra heat to warm the earth three weeks before it came to the solstice, it would give the same extra heat three weeks after; and the northern portion of the earth must continue to increase in temperature as long as the sun continues to radiate more than its medium degree of heat over the surface, at any particular place. Conversely, the whole region of country continues to grow cold as long as the sun radiates less than its mean annual degree of heat over that region. The medium degree of heat, for the whole year, and for all places, of course, takes place when the sun is on the equator; the average temperature, at the time of the two equinoxes. The medium degree of heat, for our northern summer, considering only two seasons in the year, takes place when the sun's declination is about 12 degrees north; and the medium degree of heat, for winter, takes place when the sun's declination is about 12 degrees south; and if this be true, the heat of summer will begin to decrease about the 20th of August, and the cold of winter must essentially abate on or about the 16th of February, in all northern latitudes."—p. 100.

We find some difficulty in making the different parts of this explanation seem consistent with each other; but it can hardly be necessary to tell any intelligent teacher of Natural Philosophy that no part of the explanation has any reference to the real cause of the phenomenon. May we ask how the author's theory would explain the fact that at the depth of twenty or thirty yards the temperature is highest in December and lowest in June?

In showing how to compute the equation of time, the author does not state whether the true or mean longitude of the sun is to be used in calculating that part of the equation of time which proceeds from "the variable inclination of the sun's motion to the equator." From the first explanation of the figure we should suppose he meant S , in fig. 22, p. 103, to represent the position of the mean sun; but he after speaks of φS as representing the sun's longitude, from which we might infer that he meant the true longitude to be used in finding the second correction.

On page 116 we find the statement that a planet appears stationary when moving nearly toward the earth, the author entirely forgetting that in the case supposed the planet would change its apparent position in consequence of the motion of the earth.

From page 142: "It [Mercury] is rarely visible, owing to its small size and proximity to the sun, and it never appears larger to the naked eye than a star of the fifth magnitude." Any one who has ever seen Mercury must know that it shines with nearly or quite the brilliancy of a star of the first magnitude; and it is frequently visible at its elongations, provided one takes the proper opportunity, and is not in too high a latitude.

On page 190 directions are given for finding the radius of the earth from the horizontal parallax of the moon, and the statement is made that by this means we can find the diameter of the earth to a greater degree of exactness than by any one actual measurement. The moon's horizontal parallax cannot be determined by observation to within a second of arc, that is, within about $\frac{1}{3600}$ of its whole value. Its cube is therefore subjected to an uncertainty of $\frac{1}{129600}$ of its whole value; and since the formula on p. 191 virtually contains the cube of the parallax as a factor, the final result is uncertain to the $\frac{1}{129600}$ part of its whole amount from this cause alone; and the diameter of the earth thus found may therefore be in error by six or seven miles. Yet it is said that "this result is more to be relied on than any other." We have said nothing about the uncertainty in the force of gravity, and the mass of the moon, both of which enter, or ought to enter, into the formula.

In § 186, pages 209 and 210, we read:—

"From the earliest ages the eccentricity of the orbit [of the earth] has been diminishing; and this diminution will probably continue until it is annihilated altogether, and the orbit becomes a circle; after which it will open out in another direction, again become eccentric, and increase in eccentricity to a certain moderate amount, and then again decrease.

"The period for these vibrations, though calculable, has never been calculated further than to satisfy us that it is not to be reckoned by hundreds or even by thousands of years.' It is a period so long that the history of astronomy, and of the whole human race, is but a point in comparison."

To this we reply, that it is a well-known fact that neither the eccentricity of the orbit of the earth, or of that of any of the planets, will continue to diminish till the orbit becomes circular; and it is moreover a demonstrable fact that the orbits of none of the planets will ever become mathematically perfect circles. The statement in the second paragraph is made in the face of the fact that the periods, not exactly of "these vibrations," but of the actual vibrations, have been repeatedly computed and their reciprocals published by LAGRANGE, in the "Berlin Memoirs;" by JACOBI, in his "Mathematische Werke;" by PONTÉCOULANT, in the third volume of his "Théorie Analytique du Système du Monde;" and by LEVERRIER, in the "Annales de l'Observatoire," Tom. II., and in the French "Connaissance des Temps."

On the very first page of the chapter devoted to planetary perturbations our author states that, "The direct effect of any of these bodies [the planets] on the position of the sun is absolutely insensible."

Now the actual fact is, that the effects of the attraction of the planets on the sun is always nearly as great as on each other, and sometimes so much greater that all the principal terms proceed from this cause.

"Students in Astronomy very naturally infer that the attractions of Venus and Jupiter (when those planets are nearest to the earth) affect the longitude of the moon. Indeed, some have been so confident of this as to infer that some of the lunar equations were the results of such planetary attractions. But we assure them it is not so; none of the lunar equations refer to any other disturbing body than the sun. Yet all other bodies do disturb the lunar motion, but the amount of other disturbing forces is absolutely insensible, as we shall show by the following analysis:—"

The result arrived at is, that the moon's longitude is not affected by either Venus or Jupiter by more than the "very small fraction of a second;" and the discussion, about an octavo page in length, closes with, "Therefore the perturbations produced on the moon's motion by the planets are not, and need not be, taken into account." (*Lunar Perturbations: A Fragment*, pp. 351, 352.)

Now the facts are, that some ten years ago the Astronomer Royal of England found from observation, and PROF. HANSEN from theory, that Venus produces an inequality of 20" or more in the longitude of the moon; and in HANSEN's *Lunar Tables* the sum of the terms depending on Venus may amount to nearly 40", and even a few terms depending upon Mars and Jupiter, though small, are retained. It will be seen, by any one who will take the trouble to examine PROF. PEIRCE's *Tables of the Moon*, that the effect of Venus is not omitted in the *Lunar Ephemeris*, printed in the *American Ephemeris and Nautical Almanac*. The author supposes that the problem of perturbations is solved by simply computing the intensity of the force at any instant; and we do not find even an intimation that this force is a function of the time, and that an integration is necessary to determine what change it may produce in the lapse of centuries. Indeed, the discussion shows only the first faint conceptions of the problem. But we must pass over, for want of space, numerous other and equally flagrant and unpardonable errors into which the author has fallen. That the author of a text-book on Astronomy should be so unacquainted with the modern history and present state of the science is most extraordinary; and that he should misapprehend the character of its problems is hardly less so. We do not contend that the author of an elementary treatise should be a profound and original investigator in the science of which he treats; but we do insist that he should be able to write with ordinary accuracy; that he should have an accurate knowledge of the present state of the

science; and that he should have a clear comprehension of the general principles upon which the science is based. We regret to say that our author has lamentably failed in all these necessary particulars.

Introductory Course of Natural Philosophy, for the use of Schools and Academies. Edited from GANOT's Popular Physics, by WILLIAM G. PECK, A. M., Professor of Mathematics, Columbia College, New York. Published by A. S. Barnes & Burr, 51 and 53 John Street, New York. 1860. pp. 480. 308 Illustrations.

During the past few years several excellent works on Physics have appeared in France. These works are marked by a good arrangement of the subjects treated, scientific accuracy, and clearness of statement combined with a large number of illustrations, which for beauty of design and execution have never been equalled. They are more particularly described by the term *experimental*, as distinguished from those in which the subjects are treated mathematically. On account of the acknowledged excellence of these works much use has been made of the matter and illustrations by authors preparing works on experimental Physics for the use of teachers and students in this country. Among the latest and best of them is one entitled *Cours de Physique, purement expérimentale, à l'usage des gens du monde, des aspirantes au brevet supérieur des élèves des écoles normales, des institutions de demoiselles, et en général, des personnes étrangères aux connaissances mathématiques*, par A. GANOT, Professeur de Physique; ouvrage orné de 308 magnifiques vignettes dessinées par GAGNIET, gravées sur bois par QUARTELEY, DUJARDIN, LAMBERT, et en relief, sur cuivre, par E. SALLE. We have given the title in full, to show that the author intended the work, not only for schools of certain grades, but *for the use of everybody*; and on this account, and so far as possible to avoid the necessity of apparatus, every principle is fully illustrated with copper-plate engravings far surpassing in design and execution anything ever before introduced into a text-book. We have never seen a *strictly popular work* on Physics, or Natural Philosophy, at all to be compared with it, and we sincerely thank the Editor and Publishers of the American edition for making the work more accessible to teachers and students in this country. The editor has made judicious use of the original work, and without producing an exact translation, has preserved its symmetry and better adapted it to the educational wants of this country; and the Publishers by illustrating their edition with *fac-simile* copies of the original engravings, by special arrangement with the author, M. GANOT, will have the satisfaction of knowing by its use that their efforts are appreciated. We are glad to see that the applications of fluids, steam, &c., as motive powers have not been introduced. Such an extension would have made the work too large and expensive for general use, and would not be in harmony with the plan of simply developing and explaining elementary principles. We take great pleasure in recommending this work to the attention of teachers and pupils, as well as the general reader, as being in all respects by far the best elementary one upon the subject which we have examined.

ERRATA. — Vol. III., p. 11, l. 7, for "where," read "whose"; p. 12, l. 3, for "or," read "and"; p. 14, l. 18, dele "even" after when; p. 16, last line, for $\frac{n-1}{1.2} a^{n-1} x'$, read " $n a^{n-1} x'$:" p. 17, l. 10, for $\frac{1}{16a}$ read $\frac{1}{16a^2}$.

T H E

MATHEMATICAL MONTHLY.

Vol. III . . . FEBRUARY, 1861. . . No. V.

PRIZE PROBLEMS FOR STUDENTS.

I. Find a point without two concentric circles, from which if two tangents be drawn to the circles the one shall be the double of the other.

II. Any whole number and its fifth power, when divided by 30, leave the same remainder; also, the number and its seventh power, when divided by 42, leave the same remainder. — Communicated by ASHER B. EVANS, Madison University, Hamilton, N. Y.

III. There are n straight lines making, with another fixed straight line, the angles $\alpha, \beta, \gamma \dots$; a point, P , is taken, such that the sum of the squares of the perpendiculars from it on these n lines is constant. Find the conditions that the locus of P may be a circle. — Communicated by GEORGE B. HICKS, Cleveland, Ohio.

IV. If A and B represent the semi-axes of an ellipse, the altitudes of the minimum circumscribing isosceles triangles, having their vertices in the axes produced, are as $A : B$. — Communicated by JAMES F. ROBERSON, Grantsburgh, Indiana.

V. Prove that, of all circular sectors having the same perimeter, the one of greatest area is that in which the circular arc is double the radius.

Solutions of these problems must be received by April 1st, 1861.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE
PRIZE PROBLEMS IN No. I., Vol. III.

The first Prize is awarded to A. G. BARKER, Waterville College, Me.

The second prize is awarded to H. TELMAN, Baltimore, Md.

The third prize is awarded to E. O. GIBSON, South New Berlin, N. Y.

PRIZE SOLUTION OF PROBLEM I.

By E. K. LEONARD, Marietta College, Ohio.

Let AB and CD be two diameters of a given circle, drawn at right angles to each other; AEB a circular arc described with radius DA or DB . Prove that the area of the lune $AEB C$ equals the area of the triangle ADB .

Let R be the radius of the arc AEB , and r that of the circle. When $ABC = \frac{1}{2} \pi r^2$, $ADB C = \frac{1}{4} \pi R^2$, $r : R = 1 : \sqrt{2}$. Substituting 1 and $\sqrt{2}$ for r and R , we have $ABC = ADB E$. Subtracting ABE from each, we get $AEB C = ADB$.

Most of the solutions of this problem are nearly the same as the above.

PRIZE SOLUTION OF PROBLEM II.

By E. O. GIBSON, South New Berlin, N. Y.

Of all isoperimetrical polygons having the same number of sides, the greatest is that which is equilateral.

Since an isosceles triangle has a greater area than any other triangle of equal base and perimeter, it follows that if any polygon, $ABCD$, &c., be a maximum among others of equal perimeters and the same number of sides, all the triangles, ABC , BCD , &c., must be isosceles, and have $AB = BC = CD = \&c.$

PRIZE SOLUTION OF PROBLEM III.

In any triangle, if P , P' , and P'' denote the perpendiculars from the vertices to the opposite sides; p , p' , and p'' the portions of the same between their point of intersection and the sides. Prove that

$$\frac{p}{P} + \frac{p'}{P'} + \frac{p''}{P''} = 1.$$

Mr. G. A. OSBORNE, Jr., says: Let a, b, c denote the sides of the triangle perpendicular respectively to P, P', P'' . Then

$$ap + bp' + cp'' = 2 \text{ (area of triangle),}$$

and $aP = bP' = cP'' = 2 \text{ (area of triangle).}$

Dividing each term of the first equation by the term below it, gives

$$\frac{p}{P} + \frac{p'}{P'} + \frac{p''}{P''} = 1.$$

Mr. GIBSON notices that, in an obtuse-angled triangle, as the parts of the perpendiculars let fall on the sides produced, and which are represented by p and p' , lie beyond those sides, they may be regarded as minus, and we still have

$$\frac{p}{P} + \frac{p'}{P'} + \frac{p''}{P''} = 1.$$

Mr. G. B. HICKS remarks, that this proposition may be included in the still more general statement, that, if from the angles of a triangle three lines, Aa, Bb, Cc , be drawn to meet the opposite sides, and from any point, O , within the triangle three lines, Oa, Ob, Oc , be drawn parallel to them to meet the sides, then

$$\frac{Oa}{Aa} + \frac{Ob}{Bb} + \frac{Oc}{Cc} = 1.$$

PRIZE SOLUTION OF PROBLEM IV.

By A. G. BARKER, Waterville College, Maine.

At what angle must the rudder of a vessel be inclined to the stream, that the effect produced may be a maximum?

Let the angle made by the rudder with the direction of the vessel be denoted by φ , the horizontal distance of the vessel's centre of gravity from the line of contact of the rudder by a , and the length of the rudder by b . The force exerted by the water upon the rudder parallel to the direction of the vessel's motion is proportional to $\sin \varphi$, since this force is zero for φ equal zero, and a maximum for φ

equal 90° . We may, therefore, take $\sin \varphi$ as its measure ; and since φ is the angle this force makes with the rudder, its component perpendicular to the rudder is $\sin^2 \varphi$, which we may suppose to act in the perpendicular passing through its middle point. This perpendicular produced will meet the line of direction of the vessel at a distance $\frac{\frac{1}{2}b}{\cos \varphi}$ from the line in which the rudder joins the vessel, and $a + \frac{\frac{1}{2}b}{\cos \varphi}$ is the length of the arm upon which the force $\sin^2 \varphi$ acts at an angle $\frac{1}{2}\pi - \varphi$. The component of $\sin^2 \varphi$ perpendicular to $a + \frac{\frac{1}{2}b}{\cos \varphi}$ is $\sin^2 \varphi \cos \varphi$, and

$$u = \left(a + \frac{\frac{1}{2}b}{\cos \varphi}\right) \sin^2 \varphi \cos \varphi = \max.$$

$$\begin{aligned} \therefore \frac{du}{d\varphi} &= 2a \sin \varphi \cos^2 \varphi - a \sin^3 \varphi + b \sin \varphi \cos \varphi = 0 \\ &= 3a \cos^2 \varphi + b \cos \varphi - a = 0; \end{aligned}$$

$$\therefore \cos \varphi = -\frac{b \pm \sqrt{12a^2 + b^2}}{6a}.$$

The second derivative gives

$$\frac{d^2u}{d\varphi^2} = -6a \cos \varphi \sin \varphi - b \sin \varphi,$$

which, for $\varphi < 180^\circ$, is negative for the positive value of $\cos \varphi$; and therefore this value of $\cos \varphi$ will make u a maximum.

The negative value of $\cos \varphi$ gives

$$\frac{d^2u}{d\varphi^2} = \sqrt{12a^2 + b^2} \sin \varphi,$$

a positive quantity; and u is therefore a minimum for the negative value of $\cos \varphi$.

When $b = 0$, that is, when the length of the rudder is neglected, $\cos \varphi = \pm \sqrt{\frac{1}{3}}$; $\therefore \varphi = 54^\circ 44'$, or $125^\circ 16'$.

None of the solutions of Problem V. are thought worthy of publication.

SIMON NEWCOMB,
W. P. G. BARTLETT,
TRUMAN HENRY SAFFORD.

DESCRIPTIVE GEOMETRY OF ONE PLANE.

CHAPTER I.

THE POINT, THE RIGHT LINE, AND THE PLANE.

THE POINT.

§ 1. A POINT in space is represented by its orthographic projection upon a fixed plane, called the *plane of reference*, and a number written near the projection, to show the distance of the point from this plane: this number is called the *reference* of the point with respect to the plane.

The plane of reference will at present be considered horizontal, and entirely below the magnitudes to be represented, thus avoiding any confusion which might arise from the use of negative references.

THE RIGHT LINE.

§ 2. A right line in space is determined by the projections and corresponding references of two of its points.

§ 3. PROBLEM I. *Given the projection of a point situated upon a given right line, to find the reference of the point.*

Let AB (Fig. 2) be the projection of a line, and let the references of two of its points, projected at A and B , be respectively α and β . It is required to find the reference of the point, of which C is the projection. If we imagine the line in space revolved about its horizontal projection as an axis, so as to coincide with the plane of reference, it will take the position $A'B'$, making AA' equal to α , and BB' equal to β . Calling the unknown reference of C , γ , we have

$$BA : BC = \alpha - \beta : \gamma - \beta,$$

whence

$$\gamma = \frac{BC}{BA} (\alpha - \beta) + \beta.$$

The lengths of AB and BC are known, having been measured in units of the scale (Fig. 1), and α and β are measured upon the same scale.

Conversely, if it be required to find the projection of that point of the line whose reference is γ , we have, from the same proportion,

$$BC = AB \left(\frac{\gamma - \beta}{\alpha - \beta} \right).$$

Or the point C' may be found by laying off on the line AA' a distance AD equal to γ , and drawing DC' parallel to AB .

§ 4. PROB. II. *To find the distance between two given points.*

Let A and B (Fig. 2) be the given points, then evidently the distance required will be $\sqrt{AB^2 + (\alpha - \beta)^2}$.

§ 5. *Definitions.*—The *tangent* of the angle made by the line with the plane of reference, $\left(\frac{\alpha - \beta}{AB} \right)$, is called the *declivity of the line*, and when put in the form $\frac{1}{m}$, it shows that points of the line whose projections are m units apart differ in reference by unity.

The distance between the projections of two points is called their *interval*; and the distance between the projections of two points of a line differing from each other by a unit of reference is called the *interval of the line*.

§ 6. By an application of the process explained in Problem I., it is easy to divide the projection of any given line into intervals; the projection thus divided is called the *scale of declivity of the given line*.

§ 7. PROB. III. *To construct the scale of declivity of a given line.*

Let the line be given by its projection, and the references 10.83 of the point A and 8.28 of B (Fig. 3); the interval of the points A and B , measured upon the scale (Fig. 1) is 5.75.

Therefore the declivity of the line is

$$\frac{10.83 - 8.28}{5.75} = \frac{2.55}{5.75} = \frac{1}{2.25} = \frac{1}{m}.$$

Therefore the interval of the line is $m = 2.25$. Therefore, setting off a distance 2.25 from A towards B , we have the point C , of which the reference is less by unity than that of A , i. e., 9.83.

But it is usual to divide the projection into integer references. To find the point 10, set off from A towards B a distance $2.25 \times 0.83 = 1.87$ of the scale (Fig. 1); from 10 set off a distance 2.25, which gives the point 9, &c. Divide one of the intervals, as $9 - 8$, into 10 equal parts, and the scale will be completed.

§ 8. PROB. IV. *Through a given point to draw a line parallel to a given line.*

Let the line ab be given by the projections and references of two points, 12 of a and 7.80 of b , and the point by its projection c and its reference 8.50. Draw through c a line parallel to ab , and lay off on it from c the interval ab ; this gives a point d of the required line, of which the reference is $8.50 + 4.20 = 12.7$.

THE PLANE.

§ 9. The planes which are required to be represented are of two kinds, those which are limited in extent, as faces of polyhedrons, and those which are unlimited.

An unlimited plane is determined by the position of one of its lines of greatest declivity, that is, one of the lines of the plane making the maximum angle with the horizontal plane of reference; this line is evidently perpendicular to every horizontal line of the plane so defined.

The scale of declivity of this line is called the *scale of declivity of the plane*, and to distinguish it from that of an ordinary line it is made double.

A limited plane is represented by the projection of its perimeter, to which is added either its scale of declivity or the references of its angular points. Often a plane is given by two points of its line of greatest declivity.

§ 10. PROB. V. *To pass a plane through three given points.*

Let a , b , c (Fig. 5) be the points, of which the references are respectively 10.5, 7, and 4. Join a and c , and find on ac the point d , whose reference is the same as b (7); join d and b . This will be a horizontal line of the required plane; the line of maximum declivity will be perpendicular to it.

To find another point, draw through c a line parallel to db ; it will be the horizontal (4), which determines the line of greatest declivity.

If we have the projection of any point, as e , situated upon this plane, to find its reference, it will be sufficient to drop a perpendicular from e to the scale, and find by Prob. I. the reference of the point of intersection, which will be the reference required.

§ 11. PROB. VI. *To find the intersection of two planes.*

Let the two planes M and N (Fig. 6) be given by their scales of declivity.

In one plane, M , draw any two horizontals; through 7 and 11, for instance. In the other plane, N , draw horizontals having the same references, and the points a and b , their intersections, are two points of the line of intersection of the two planes.

It may happen that the horizontals of each plane are nearly or quite parallel. In the first case, take two auxiliary planes, Q and P (Fig. 7). Q intersects M in the line xy , and also N in the line vt ; therefore a , the intersection of xy and tv , is a point common to the three planes M , N , and Q , or a point of the line of intersection of M and N . The same construction with P gives the point b ; whence ab is the line of intersection required.

If the horizontals were exactly parallel, one auxiliary plane would be sufficient, since the line of intersection would be a horizontal.

§ 12. PROB. VII. *Through a given point in a given plane, to draw a line of the plane having a given declivity.*

Let M (Fig. 8) be the given plane, a the projection of the given

point, and 2.7 its reference; let the declivity of the required line be $\frac{1}{8}$. Draw any horizontal of M , as 6, and take the difference between its reference and that of the given point ($6 - 2.7 = 3.3$); multiply this difference by the reciprocal of the declivity of the required line ($3.3 \times \frac{1}{8} = \frac{1}{2} = 5.5$); then with the projection of the given point as a centre, and a radius 5.5, describe an arc cd ; and where it meets the horizontal 6 will be another point of the required line.

There will be two solutions, if the declivity of the line be less than that of the plane; one, if it be equal, none, if it be greater.

§ 13. PROB. VIII. *Through a given line pass a plane of a given declivity.*

Let the declivity of the required plane be $\frac{1}{m}$ (Fig. 9). Conceive any point of the line, as b , to be the vertex of a cone, of which the elements have a declivity $\frac{1}{m}$; to trace the base of the cone, it will be sufficient to take a radius, m , and with b as a centre describe a circle; the plane of this base will be horizontal, whose reference is less by unity than that of b . Now, from the point c upon the given line, whose reference is less than b by unity, draw a tangent to the circle; this tangent will be a horizontal of the required plane. Draw a parallel through b , and we have another, whence its line of declivity is immediately determined. In general there will be two solutions.

Sometimes it happens, in the solution of this problem, that the line being very little inclined, the position of the second point would lie off the paper. The solution for this case is to take any other point, d , of the line, and consider it also the vertex of a cone having its base on the same horizontal plane, and its declivity the same, the radius of its base would be m times the altitude; a horizontal of the plane would then be found by drawing a tangent common to the two circles of the base. (Fig. 10.)

If the line given were exactly horizontal, then one cone would be sufficient.

§ 14. PROB. IX. *To find the point in which a line pierces a plane.*

Let M (Fig. 11) be the given plane, and A the given line. Through the line A pass any arbitrary plane, X , and find its intersection, mn , with M ; the point in which this line of intersection meets A is the point sought.

This method may be used to ascertain the point in which two lines having the same projection intersect.

Pass through the two given lines any two arbitrary planes, X and Y , and determine their intersection; the point in which this intersection meets the given lines is their common point. Or the point of intersection may be found by revolving the lines about their common projection as an axis until they are parallel to the horizontal plane of reference. [The projection may be considered as situated in any horizontal plane, since, as in this case, it is often convenient to work upon horizontal planes of considerable references.]

Fig. 12 represents these two solutions.

§ 15. PROB. X. *From a given point in space to drop a perpendicular upon a given plane, to find the point in which it pierces the plane and the length of the perpendicular.*

First Solution. — The projection of the perpendicular will evidently be parallel to the scale of declivity of the plane, and its declivity the reciprocal of that of the plane.

Suppose the declivity of the plane is $\frac{1}{m}$, then the interval upon the perpendicular will be $\frac{1}{m}$.

Therefore, through the projection of the given point a draw a line parallel to the scale of declivity of the plane, and lay off from a on this line intervals of $\frac{1}{m}$, measured upon the scale of equal parts (Fig. 1); the references of the perpendicular will increase in a direction contrary to that of the scale of declivity of the plane.

The point in which the perpendicular pierces the given plane may be found by § 16, and its length by § 4.

If the line of declivity of the plane be given by the references of two of its points, c and d , the perpendicular can still be drawn without constructing the scale of the plane. For after drawing the parallel to the line of declivity of the plane, set off on it from the point a a distance equal to the difference of the references of c and d , and the reference of this last point will be the reference of a *plus* or *minus* the length of the interval between c and d , according as it is laid off *above* or *below* the point a . That is, intervals of the plane correspond to differences of reference on the perpendicular, and reciprocally.

There is another solution to this problem, which may be used to advantage in many of the subsequent problems.

Let a (Fig. 13) be the given point, $d e$ a section of the plane of reference, $d c$ the line of maximum declivity of the given plane, $a c$ the required perpendicular, and $d e$ its horizontal projection; $\tan c d e = \frac{1}{m}$, the declivity of the given plane. Then the length of

$$a c = a d \cos \tan^{-1} \frac{1}{m}, \quad \text{and} \quad d e = a d \sin \tan^{-1} \frac{1}{m} \cos \tan^{-1} \frac{1}{m};$$

these two may be readily constructed in connection with the scale of declivity of the plane.

Let M (Fig. 14) be the given plane, and a be the projection of the point of which the reference is 11. Let fall from a a perpendicular to the scale, and find the corresponding reference 8.4 of b . From b set off a distance $b c$ equal to an interval of the scale M , and also set off $b d$ equal to a unit of the scale of equal parts (Fig. 1); join c and d , and produce this line. From c set off a distance $c e$ equal to 2.6, the difference of reference between b and a . Draw $e f$ perpendicular to $c d$, and $f h$ parallel to $c e$; and the point h , where the last line cuts the parallel through a to the scale M , is the point sought. $a h$ is the

projection of the required perpendicular and cf is its length; the angle fce is the angle of declivity of the plane.

The interval ah may be easily calculated. Let the declivity of the plane be $\frac{1}{m}$, and the difference of reference between the points a and b be denoted by δ ; then the length of the required perpendicular is

$$p = \delta \cos \tan^{-1} \frac{1}{m} = \delta \cdot \frac{m}{\sqrt{m^2 + 1}},$$

$$ah = \delta \cdot \cos \tan^{-1} \frac{1}{m} \sin \tan^{-1} \frac{1}{m} = \delta \frac{m}{\sqrt{m^2 + 1}} \cdot \frac{1}{\sqrt{m^2 + 1}} = \delta \frac{m}{m^2 + 1}.$$

The difference of reference between the point h and the horizontal ab is $\delta \sin^2 \tan^{-1} \frac{1}{m} = \delta \frac{1}{1 + m^2}$.

§ 16. PROB. XI. *To draw through a given point a plane perpendicular to a given line.*

The projection of the scale of declivity of the plane must be parallel to that of the given line, and its declivity is the reciprocal of that of the line.

So the scale is found in the same manner as that of the required line by the first method of the last problem.

§ 17. PROB. XII. *To find the shortest distance from a point, a , in space to a line, A .*

First Method. — Pass through a a plane perpendicular to A , and find the point in which A pierces this plane; the line connecting this last point with a will be the perpendicular required.

Second Method. — Let a (Fig. 15) be the given point, and A the given line; join A and the point b of A having the same reference (7), ab will be a horizontal of their plane (X); rotate this plane about ab as an axis until it becomes parallel to the horizontal plane of reference; any point of A , as c , will be found at c' , by taking sc' equal to the distance between the axis and c in space. Therefore A' will be the revolved position of A ; ac' will be the length of the

perpendicular, and, making the counter revolution, we have $a d$, its projection.

§ 18. PROB. XIII. *Through any two lines, A and B , pass two parallel planes.*

Through any point of A draw a line parallel to B ; these two lines will determine the first plane. Then through any point of B draw a plane parallel to the first.*

§ 19. PROB. XIV. *Through a given line, A , to pass a plane perpendicular to a given plane, M .*

From any point of A drop a perpendicular upon M ; this perpendicular and the line A determine the required plane.

§ 20. PROB. XV. *To measure the angle of two lines, A and B .*

Draw through any point of A (Fig. 17) a line parallel to B , as $a y$, and pass a plane through A and $a y$. Rotate this plane about any horizontal $x y$, so as to be parallel to the plane of reference. The new positions of the lines determine the angle sought, $x a' y$.

Since the three sides of the triangle $x a y$ are very easily determined, the angle may be found without rotation by the formula,

$$\cos A = \frac{b^2 + c^2 - a^2}{2 b c}.$$

§ 21. PROB. XVI. *Through a given point, a , to draw a line which shall make a given angle, μ , with a given line, A .*

Join a (Fig. 18) and the point of A having the same reference, this will be a horizontal of their common plane; rotate this plane till it is parallel to the plane of reference, and find A' , the new position of the line A . Draw through a a line $a c'$, making the given angle μ with the line A' , and make the counter revolution: $a c$ will be the line sought.

If the tangent of the angle μ be given, $\frac{1}{m}$, the point c' may be

* The construction for drawing one plane parallel to another is the same as for drawing one line parallel to another, and was explained by Prob. IV. Fig. 16 gives the construction.

found by dropping from a the perpendicular ad' , and measuring off the distance $d'e'$ equal to m times ad' .

§ 22. PROB. XVII. *To construct the angle of a given line, A , and a plane, M .*

From any point of A drop a perpendicular upon M , and by Prob. XV. find the angle between A and this perpendicular; this will be the complement of the angle sought.

§ 23. PROB. XVIII. *To find the scale of declivity of a line, knowing its projection, and also knowing it to be perpendicular to a given line, A , at a given point, a .*

Through the angular point a draw a plane, X , perpendicular to the given line. The unknown line will be in this plane; therefore drawing in X any horizontal, the point in which it meets the given projection of the unknown line will correspond to a point of the required line of the same reference; whence its scale may be immediately constructed.

§ 24. PROB. XIX. *To find the scale of declivity of a line, knowing its projection, and also the angle, μ , which it forms with a given line at a given point.*

If with the given angular point as a centre a sphere be conceived to be described, the four points of its surface in which the two lines and their two projections pierce it will form the angular points of a spherical quadrilateral, a, b, c, d (Fig. 19), in which ab is the angle of declivity of the given line, ad the given angle μ , cb that of the projections, abc, bcd are right angles, and dc is the unknown angle of declivity: whence the side dc may be calculated by the rules of spherical trigonometry; but the graphic solution is as follows:—

Let a (Fig. 20) be the angular point, ab the projection of the given line, ae that of the unknown line. Pass a plane through the given line and the projection of the unknown line; its scale will be M , its angle of declivity Mfg , the complement of which is gfe . Rotate

the line ab about ae as an axis, so as to coincide with the horizontal plane through a ; it will take the position ac .

Make the angle cad equal to the given angle μ . Now, comparing Figs. 19 and 20, we have the three parts of the spherical triangle acd (Fig. 19), ac , acd , and ad , which correspond respectively to ea , gfe , and cad of (Fig. 20); and it is required to find the side dc (Fig. 19), or, in other words, the other face of the pyramid. To do this, erect at any point m of ae (Fig. 20) a perpendicular, mn ; make the angle mnr equal to gfe ; then mr is a line of the unknown face revolved about nm as an axis, to coincide with the horizontal plane. At n erect a perpendicular to mn , and also draw nv perpendicular to ac , and ns perpendicular to nv and equal to nr . The line nvx is the horizontal trace of a vertical plane; and joining s and x , sx is the revolved position of the intersection of the unknown face and the vertical plane ux . With v as a centre and uv as radius describe an arc ut , cutting sx in t . Let fall from t the perpendicular tz to ux , and from z a perpendicular, zy , to ae . With x as a centre, and a radius xt , describe an arc meeting zy produced in t' . Join t' and a , and we have eah , the angle of declivity sought, and ah , the position of the line required revolved about its horizontal projection.

§ 25. PROB. XX. *To reduce an angle to the horizon; that is, having given μ the angle which a line forms with a given line at a certain point, and also its declivity, to find its projection.*

Let ab (Fig. 21) be the given line, and a the angular point. Revolve the line about its projection so as to coincide with the horizontal plane through b ; it will take the position ba' . Make the angle $aa'c$ equal to the angle of declivity of the unknown line, and $ba'd$ equal to μ , the given angle.

$a'e$ intersects ab in c , therefore with a as a centre, and a radius ac , describe a circle; it will be the base of a cone of which a is the vertex, and every element of this cone will have the given declivity.

Take $a'd$ equal to $a'c$, and with b as a centre and bd as a radius describe an arc intersecting the former in the points m and n ; am or an will be the projection required. The line required and the given line are represented in revolved position by $b a' d$, and the third side of the triangle $b a' d$ is horizontal.

§ 26. *Transformation of the Plane of Reference.* — It is convenient often to change the plane of reference and refer the magnitudes to another plane whose scale with respect to the primitive is given; this may be done by an application of Prob. X.

I. *To change with respect to a point, i. e. to find the new projection and reference of a given point, a , with respect to any plane, M .*

Let fall from a a perpendicular upon M ; find the point a' , in which it pierces the plane M , and the length of the perpendicular.

The point a' is the new projection, and the length of the perpendicular is the reference sought.

II. *To change with respect to a line.*

Find the new projections and references of two points of the line. The construction may be simplified by taking as one point that in which the line pierces the plane M , which is its own projection, and has a reference zero.

III. *To change with respect to a plane; change with respect to any three points of the plane.*

The construction is much simplified by finding first the line of intersection of the given plane and the new plane of reference; this line is the new horizontal of zero;* dropping upon this intersection a perpendicular from the new projection of another point of the plane, this will be the projection of the new scale, which is known by the references of two points.

The scale of the primitive with respect to the new plane of refer-

* The term horizontal here is evidently a misnomer, but it cannot lead to any ambiguity.

ence will have the same projection and interval, but will be the negative of it, *i. e.* it will increase from zero in the opposite direction.

The position of the new plane of projection may not be given directly, it may be required to be parallel or perpendicular to a given line or plane. The Problems IV., X., and XI. enable us to find immediately the scale of declivity required.

§ 27. PROB. XXI. *To find the angle of two planes, M and N .*

First Method.—Take the plane M as a new plane of reference; the new angle of declivity of N will be the angle sought. Fig. 22 gives the construction. Find the intersection ce of the two planes M and N ; this is the new horizontal. Let fall from any point of N , as a , a perpendicular upon M ; it pierces it at a' , and its length is bd . Let fall from a' a perpendicular, $a'f$, to ce , the new horizontal; then $a'f$ is the line of declivity of N with respect to M . Erect at a' a perpendicular to $a'f$ equal to bd , the new reference of a' ; join g and f ; fg is the revolved position of the line of maximum declivity of N , and $a'fg$ is the angle required.

Second Method.—Let fall from any point in space, a , a perpendicular upon each plane, and measure the angle of these perpendiculars by § 20; this will be the supplement of the angle required. Or the length of the perpendicular upon either plane, as M , divided by the distance from its foot to the point in which the perpendicular upon N pierces the plane M , will be the tangent of the angle sought.

Third Method.—Let M and N (Fig. 23) be the two planes, ab their line of intersection. Conceive a plane drawn perpendicular to ab at the point y ; xyz will be a horizontal of this plane, having a reference 70. From y let fall a perpendicular, yy' , upon ab' , the revolved position of the line of intersection of the planes, and take yy'' equal to yy' , the angle $xy''z$ will be the angle sought.

§ 28. PROB. XXII. *Through a given right line to pass a plane which shall make a given angle, μ , with a given plane.*

Let A (Fig. 24) be the given line, and M the given plane. Take M as the new plane of reference.

To find the new position, A' , of A , find first the point a , in which A pierces M ; and then from any point of A , as d , let fall a perpendicular upon the plane M ; it pierces this plane at the point c , and its length cx is the new reference of c ; therefore ac is the new projection A' . Denote $\tan \mu$ by $\frac{1}{m}$, the given declivity. Conceive any point of A' , as k , the vertex of a cone, whose elements have the given declivity. The base of this cone is a circle situated in the new plane of reference, having for a centre k , and for a radius the reference of k multiplied by m .

Drawing from a a tangent to this circle, it will be a horizontal of the required plane with respect to the new plane of reference. This new horizontal meets the horizontal of zero of the primitive plane at the point y , therefore y is common to the primitive plane, to the new plane, and to the required plane; it is also its own projection in both planes, and has a reference zero. Therefore, joining y and the point z of the line A , whose reference is zero, we have the line zy , a horizontal of the required plane with respect to the primitive plane. Any other parallel, as through 40, is sufficient to determine the scale of declivity N of the required plane. The problem has in general two solutions. The other plane is determined by the other tangent, ay' , and has for a scale N' .

Second Method.—Let $\frac{1}{m}$ be the tangent of the given angle, A the given line, and M the given plane. From any point of A let fall a perpendicular upon M , and measure its length, p . Measure also the distance from the foot of p to the point in which A pierces M , and call this distance A' .

The distance from the foot of p to the intersection of the unknown plane with m is $m \cdot p$, and the sine of the angle which A' makes with the same intersection is $\frac{m \cdot p}{A'}$.

Through the foot of p pass a plane perpendicular to A' ; and upon its intersection with M , starting from the foot of p , lay off a distance equal to $A' \times \tan \left(\sin^{-1} \frac{p \cdot m}{A'} \right)$, or $\frac{A \cdot p \cdot m}{\sqrt{(A'^2 - p^2 \cdot m^2)}}$; and this point will be one of the required planes which, with the given line, immediately determines it.

§ 29. PROB. XXIII. *To determine the perpendicular between two right lines not in the same plane.*

First Solution. — Let A and B (Fig. 25) be the two lines. Pass a plane, X , perpendicular to A at any point, c , and take it as the new plane of reference. The new projection of B is B' , and that of A is reduced to the point c . Draw cr perpendicular to B' ; this will be the perpendicular required. Since cr in space is parallel to the plane X , the new reference of c (the point in which cr meets the new projection of A) is the same as that of r . Therefore, returning to the primitive plane of reference, the point r is projected in t , and making cs in space equal to the new reference of r , st will be the projection of the perpendicular between the lines A and B , and cr will be its length.

Second Direct Solution. — Let A and B (Fig. 26) be the given lines. Pass through B a plane, X , parallel to A . Then from any point of A , as a , let fall a perpendicular to X ; it pierces the plane X in a' , and ca' is its length. A line through a parallel to A will be the projection of A upon the plane X . The point n , in which this projection meets B , is one point of the perpendicular sought; drawing mn parallel to aa' , we have the required projection of the perpendicular distance between A and B .

§ 30. PROB. XXIV. *Given a plane, M , and the projection of a limited line, A , of this plane to construct upon A a cube which shall rest upon M .*

Let ad (Fig. 27) be the length, A , of the given projection. Make the angle sxy equal to the angle of declivity of M . Rotate the plane M about any horizontal x , as 50. To find the new position of

d , let fall from d a perpendicular to $s x$, and produce it to n on the line $x y$; then $x n$ will be the distance of d from the axis of rotation, and will make known the side $a d'$ of the cube. The square $a d' c' b'$ will be the base in revolved position, making the counter rotation, the base will be projected in $a b c d$. To find the upper base after revolution, make $x y$ equal $a d'$, and let fall the perpendicular $y s$; this will be the length of the horizontal projections of the edges of the cube projected at $a b c d$; whence, drawing through a, b, c, d lines perpendicular to the axis of rotation and equal to $y s$, we have the upper vertices; and the difference of reference between any two points symmetrically situated, as b and h , is $s x$, whence the cube is completely determined.

§ 31. PROB. XXV. *To find the intersection of two polyhedrons.*

Solution. — Pass a number of horizontal planes through each solid. Then the intersections of horizontals of the same reference, drawn in the intersecting adjacent faces of the two polyhedrons, make known the lines of intersection of those faces.

Fig. 28 represents an application of this method to the determinations of the intersections of two quadrangular pyramids.

§ 32. PROB. XXVI. *To find the shadow of any polyhedron.*

Let $a b c d e$ (Fig 29) be the lower base, and $a' b' c' d' e'$ the upper base of a polyhedron. Let the reference of the upper base be 80, and that of the lower base 0. Instead of supposing, as usual, the rays of light parallel to the diagonal of a cube resting upon the horizontal plane, let them be supposed to make an angle of 45° with the plane of reference, and have a direction indicated by the arrow A .

To find the shadow of any point, as f' , draw through f' a parallel to A , and lay off a distance $f' f''$ equal to the reference 80 of f' . The same constructions for d' and e' give the points d'' and e'' . Joining d and d'' , f and f'' , we have $f f'' e'' d'' d$ as the shadow required.

NOTES AND QUERIES.

I. *A General Method of finding Tangents to Algebraic Curves.*— If the curve be referred to rectangular axes, and $x_1 y_1$ be the co-ordinates of the point of tangency, the equation of a line passing through this point will be of the form,

$$y - y_1 = a(x - x_1).$$

It is required to find the value of a when this line is tangent to the curve. Refer the curve to a system of polar co-ordinates, whose pole is at the point $x_1 y_1$, and axis parallel to the axis of x . Since the pole is upon the curve, the absolute terms will disappear from this equation. Every term of the equation will now contain r , the first power of which may be divided out. Then making $r = 0$, find the corresponding value of $\tan \theta$, which will evidently be the required value of a . This value of $\tan \theta$ may always be found by the solution of an equation of the first degree. For any term that contains the product of $\sin \theta$ and $\cos \theta$, or the higher powers of either, will also contain the higher powers of r , and will therefore disappear when $r = 0$. Hence it readily appears that in practice we may neglect all terms except those which contain the first power of r ; and from these we shall find the required value of $\tan \theta$.

This method will readily suggest its own demonstration.

EXAMPLES.

1. Take the folio of Descartes, of which the equation is

$$x^3 - 3cx y + y^3 = 0.$$

The equations of transformation are

$$x = x_1 + r \cos \theta, \qquad y = y_1 + r \sin \theta.$$

Hence we have

For numbers ending in 25 and 75 let h represent the hundreds, then 25 and 75 will be represented by $\frac{1}{4}$ and $\frac{3}{4}$; annexing each of these fractions to h , we have the two binomials $(h + \frac{1}{4})$ and $(h + \frac{3}{4})$; squaring each and factoring, they become

$$[h(h + 5) + \frac{1}{16}], \quad [(h + 1)(h + 5) + \frac{9}{16}],$$

each of which may be readily expressed in the form of a rule, by observing that the square of numbers ending in 25 or 75 always ends in 625. Thus,

$$(225)^2 = 50625, (725)^2 = 525625, (375)^2 = 140625, (975)^2 = 950625.$$

Let $h = 100$ and $t = 10$, then we have, for the powers of the above numbers,

$$\begin{aligned} (225)^2 &= (\frac{9}{4} h)^2 = \frac{81}{16} h^2 = 50625, \\ (375)^2 &= (\frac{15}{4} h)^2 = \frac{225}{16} h^2 = 140625, \\ (225)^3 &= (\frac{9}{4} h)^3 = \frac{729}{64} h^3 = \frac{729}{8 \times 8} h^3 = \frac{91125}{8} = 11390625, \\ (25)^4 &= (\frac{5}{2} t)^4 = \frac{625}{16} t^4 = 390625, \\ (15)^6 &= (\frac{3}{2} t)^6 = \frac{729}{64} t^6 = \frac{729}{8 \times 8} t^6 = \frac{91125}{8} = 11390625. \end{aligned}$$

Numbers may also be multiplied together in the same manner. Thus,

$$\begin{aligned} (15 \times 25 \times 5) &= (\frac{3}{2} t \times \frac{1}{4} h \times \frac{1}{2} t) = \frac{3}{16} t^2 h = \frac{3}{16} h^2 = 1875, \\ (35 \times 75 \times 65 \times 5) &= (\frac{7}{2} t \times \frac{3}{4} h \times \frac{13}{2} t \times \frac{1}{2} t) = \frac{273}{8 \times 4} t^3 h = \frac{34125}{4} = 853125. \end{aligned}$$

— JAMES F. ROBERSON, State University, Bloomington, Indiana.

III. *Solution of Problem V., No. X., Vol. II.* — Suppose the sphere to be divided into an infinite number of concentric spherical shells; let A be their common centre, and B the exterior point; and M any particle on the surface of one of these shells, $r = AM$, $c = AB$, $y = BM$, dr = the thickness of the shell, and ρ = its density.

Since dr is infinitely small, ρ may be considered constant for the

same shell, but varying for *different* shells. The resultant attraction of the whole shell on

$$B = \frac{\pi \rho r dr}{c^3} \int_{c-r}^{c+r} \left(1 + \frac{c^2 - r^2}{y^3}\right) dy = \frac{4 \pi \rho r^3 dr}{c^3}.$$

To obtain the part of the shell which exerts half this attraction, take $c - r$ and x for the limits of y , and put

$$(1) \quad \frac{\pi \rho r dr}{c^3} \int_{c-r}^x \left(1 + \frac{c^2 - r^2}{y^3}\right) dy = \frac{2 \pi \rho r^3 dr}{c^3}.$$

But $\int_{c-r}^x \left(1 + \frac{c^2 - r^2}{y^3}\right) dy = x - \frac{c^2 - r^2}{x} + 2r$; therefore equation (1) gives $x - \frac{c^2 - r^2}{x} + 2r = 2r$, or

$$(2) \quad x = \sqrt{(c^2 - r^2)}.$$

If C be on the surface of the shell, and $BC = x$, the triangle ABC will be right angled at C . If we suppose r in equation (2) to take successively all values between zero and the radius of the sphere, the triangle ABC will be right angled in every case; hence the locus of the point C , which is always on the bisecting surface, is the surface of the sphere described on AB as diameter. — ASHER B. EVANS, Madison University, Hamilton, N. Y.

IV. *On an Approximate and Graphical Rectification of the Circle.** — The following treatment of this oft-attempted problem is founded upon the singularly close approach that is made by the angle $\tan^{-1}(\frac{1}{4}\pi)$ to a root of the equation $\sec x = \cot x$. The tabular solution of this equation is presented in the form

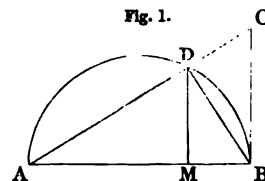
$$\cos x = \tan x = \sqrt{\left(\frac{\sqrt{(5)} - 1}{2}\right)} = 0.7863,$$

while $\frac{1}{4}\pi = 0.7854$; so that $x = 38^\circ 10' 46''$, and $\tan^{-1} \frac{1}{4}\pi = 38^\circ 8' 46''$:

* From the London Quarterly Journal of Pure and Applied Mathematics, October, 1860.

from which it is clear that if a simple construction can be found for obtaining the angle x that satisfies this equation, the tangent of that angle will yield a very close approximation to the desired periphery of a circle.

Let the chord ADC (Fig. 1) cut the semicircle ADB in D , and the tangent BC in C , in such a manner that $AD = BC$; then shall AD , BC represent *quam proximé* quadrants upon the semicircle ADB . By similar triangles, $\frac{AD}{AM} = \frac{AB}{AD} = \frac{AB}{BC}$, by hypothesis. Or $\sec DAB = \cot DAB$, and DAB is such an angle as x ; therefore



$$BC = AB \tan DAB = 2 \tan x, \text{ if the radius be unity;} \\ = 1.5727; \text{ and } \frac{1}{2} \pi = 1.5708.$$

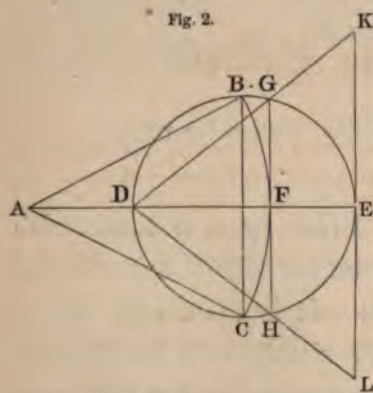
Therefore BC exceeds the quadrant of ADB by one thousandth part nearly; and in consequence, the triangle ABC exceeds the semicircle ADB by one thousandth part. It is upon such a singular coincidence as this that the following simple rule for approximate rectification of the circle with compass and ruler has been framed.

Rule.—“From an external point, at a distance equal to the radius from the given circle, describe a circle cutting the given circle at the two extremities of that diameter which is perpendicular to the line joining the point with the centre of the circle; and perpendicular to the same line draw two tangents to the two circles, the first touching the original circle at the point most remote from the external point, and the second touching the other circle, and cutting the original circle in two places. Then the two chords joining these two last points with the point of the given circle nearest to the external point, will represent the two quadrants that are there terminated; and if these chords be produced until they meet the tangent first

drawn, they will intercept upon it a portion bisected at the point of contact, and representing the two quadrants that are there terminated: and the triangle which this tangent forms with the two chords so produced will represent the area of the circle: and each of these rectilinear quantities will exceed the corresponding curvilinear quantity which it approximately represents by one thousandth part of the whole, nearly."

In proof of this construction, it will be seen that both DG and EK (Fig. 2) have by the construction the numerical values

Fig. 2.



$\sqrt{[2 \{ \sqrt{5} - 1 \}]}$, radius being unity, and are therefore equal to one another, whence the inference proceeds as before.

This construction closely rivals in descriptive geometry the well-known arithmetical approximation to the value of π that is found in the vulgar fraction $\frac{22}{7}$, this fraction exceeding the true value by only four, and the above geometrical result exceeding the same by only twelve parts in ten thousand. This simple process will, for example, yield for the quadrant of a circle ten inches in diameter, a result exceeding the truth by less than the hundredth part of an inch, which amply suffices in accuracy for all that such processes can have in view.

P. S. — It has been pointed out to me that the equation $\sec x = \cot x$, which the angle $\tan^{-1} \frac{1}{4} \pi$ so nearly solves, is equivalent to that proposition which Herodotus has recorded concerning the dimensions of the great pyramid of Gizeh, when he says that the area of the sloping side is equal to the square upon the perpendicular height. The semiangle contained at the vertex between two opposite sloping sides is in this case such an angle as x , and

accordingly the above construction may be considered as affording a ready means for reproducing upon paper those proportions which a recent writer* has affirmed that this monument was intended to perpetuate, the periphery of its base bearing to its height very nearly the proportion of the circumference of a circle to its radius. The triangle AKL in the above construction will accordingly represent the vertical section of the Great Pyramid parallel to its sides. —
A. S. HERSCHEL, Collingwood, England, August 18, 1860.

Mathematical Monthly Notices.

General Problems from the Orthographic Projections of Descriptive Geometry; with their Applications to Oblique — including Isometrical — Projections, Graphical Constructions in Spherical Trigonometry, Topographical Projection ("One Plane Descriptive"), and Graphic Transformations. By S. EDWARD WARREN, C. E., Professor of Descriptive Geometry and Geometrical Drawing in the Rensselaer Polytechnic Institute, Troy, N. Y. New York: John Wiley, 56 Walker Street, 1860. 8vo. pp. xxxv, 412.

The aims of PROFESSOR WARREN in this work are thus stated in his Preface: —

"1st. To proceed upon the basis of a perfectly accurate and rational idea of what Descriptive Geometry is.

"2d. To make the course perfectly *determinate* and rational, even to the very details, in the plan of its composition and the order of its progression.

"3d. To embody in its methods of treatment, as well as in its preparatory directions to the student, safeguards against false modes of study and sham scholarship.

"4th. To make the course sufficiently full to meet the wants of advanced students, and of the members of the professions based on mathematics and the physico-mathematical sciences."

The plan of the course is founded on the division of Surfaces into Ruled Surfaces and Double-curved Surfaces, and the subdivision of Ruled Surfaces into Plane Surfaces, Single-curved Surfaces, and Warped Surfaces. Upon each of these four classes of surfaces four operations are distinguished, namely, Projections, Intersections, Tangencies, and Developments.

In carrying out this plan the *General Problems* are placed under the two heads of *Lower Descriptive Geometry* and *Higher Descriptive Geometry*. Under the first head are treated quite fully the various problems relating to the plane, straight line, and point; those relating to the cone and cylinder, including conic sections; those relating to the hyperboloid of one nappe, and the hyperbolic paraboloid; and lastly, those relating to the sphere, ellipsoid, elliptic paraboloid, and hyperboloid of two nappes. Under the second head we find the construction of various spirals, the cissoid, conchoid, involutes, cycloids, helix, and spherical epicycloid; also problems on the developable helicoid, warped oblique arch, oblique helicoid, right conoid, and annular torus.

* "The Great Pyramid, and why it was built," by JOHN TAYLOR.

Book II. gives the *Applications of Orthographic Projections*. These include Isometrical Projections, Conventional Oblique Projections, Spherical Trigonometry, Topographical Projections (a name which the author prefers to One-plane Descriptive Geometry), and Graphic Transformations, which include changing the planes of projection and rotating given magnitudes. A collection of Miscellaneous Problems, sixty-three in number, closes the work.

From the above it will be seen that PROFESSOR WARREN's book is far more comprehensive than any work on Descriptive Geometry before offered to American, or, so far as we know, to English readers. As such it will doubtless be welcomed by the Professors in our Scientific Schools, and by many students to whom the extensive French works on this subject are inaccessible. But, besides its comprehensiveness, this book has many positive merits. The author appears to have studied his subject well, and to be master of its minutest details. He is without doubt familiar with the works of his predecessors; but the marks of independent thought are apparent on even a cursory examination. His experience as a teacher has enabled him to judge of the wants of students, and to make many useful suggestions in the shape of remarks. We find also several examples of extended discussions of problems, that show a praiseworthy love of thoroughness.

With this general recognition of the merits of the book before us, we pass to some more particular remarks upon a few points.

The true sphere of Descriptive Geometry is one of the topics discussed in the Preface, and the conclusion arrived at is, that Descriptive Geometry is "the *exact graphic art*, and not the science of geometry, and is distinctively, *graphically operative geometry*, or the *Geometry of Graphic Problems*." In this the author agrees with CHARLES (LIOUVILLE'S *Journal*, 1847, p. 33), who also denies to Descriptive Geometry the name of a science. But there are many able supporters of the opposite view, among whom may be noticed MONGE, CARNOT, PONCELET, and OLIVIER. M. OLIVIER discusses the subject quite fully in the preface to his *Additions au Cours de Géométrie Descriptive*, and becomes quite indignant against those who deny the claims of Descriptive Geometry to be called a science as well as an art.

PROFESSOR WARREN has also some remarks on the use of models and pictorial diagrams in teaching Descriptive Geometry, and condemns their general use, on the ground that without them certain powers of the mind — attention, abstraction, and especially conception — are better cultivated. This objection would be more forcible, if these powers could be cultivated in no other way, or if there were not many persons to whom Descriptive Geometry would be very useful, if it could be brought within their grasp without long and to them painful study. The truth is that no real simplification of any science is to be dreaded, even when we regard study as a discipline of the mind. When fields once inaccessible to ordinary minds are annexed to the common domain, there will always be higher fields opening up to exercise and strengthen the loftiest faculties.

OLIVIER, with an experience of more than twenty years in teaching Descriptive Geometry, recommends the use of models in the strongest terms. The costly models bearing his name are well known; but he has also contrived a simple apparatus which any one can have made for himself, and which we shall here describe for the benefit of those who have not access to this author's works. Take two shallow boxes similar to the tables of a backgammon board, and connected like them by hinges. Fill up these boxes with sheets of cork, and attach a brace to hold one table at right angles to the other, when necessary. These are the planes of projection. Provide four sets of rods from one eighth to one fifth of an inch square, and of various lengths from four inches to twelve, or sixteen, or twenty four inches, or even longer, according to the size of the planes. These rods are to be pointed with needles, that they may be readily fixed in any position in the cork. They are to be painted, one set red, one red and white in

bands, one black, and one black and white in bands. The red rods are used to construct in space any system referred in position to the two planes of projection; the red and white rods to project the points of this system on each of the planes of projection; the black and white to project these projections on the ground line; and the black to represent the projections of the red lines in space. When by means of these rods the operations necessary for the solution of a problem have been completed, the red and the red-and-white rods are removed, and the vertical plane revolved on its hinges to a horizontal position. The work is then done. By this apparatus also all ruled surfaces, such as cones, cylinders, hyperboloids of one nappe, hyperbolic paraboloids, conoids, &c., may be exhibited.

The classification of subjects adopted by PROFESSOR WARREN will arrest the attention of every one. Few, we think, will approve it. System is in itself a great good; but here the dividing, subdividing, and dividing again, are carried so far, that one almost doubts whether the confusion incident to so many heads is not greater than that of no classification whatever. A few distinct heads, such as those generally adopted by writers on this subject, are all that are necessary, at least in a text-book.

The style is also frequently open to objection. The greatest simplicity, directness, and precision should characterize such a work. All long and wordy sentences, all parade and circumlocution, all vagueness and ambiguity, are serious hindrances to the student. As an example of prolixity (one of the worst perhaps), we make an extract from the chapter on Graphic Transformations.

"426. Beneath all the formal problems of Descriptive Geometry, and giving character to their mode of solution, is a tacitly recognized problem, founded in the failure of human inventive skill to keep pace with human knowledge, as seen in the incapacity, hitherto apparent, to make *available*, that is, *cheap* and *permanently reliable*, instruments for the *easy* and *exact* graphical construction, by a continuous movement, of any lines but right lines and circles, — other lines being usually determined by points which are afterward joined by hand, or by the use of the 'irregular curve' (ruler).

"427. In continuation of this reasoning, it appears: 1st. That any point revolves about a fixed axis in a circular arc; 2d. In order to secure an adequate, that is, an immediately available, representation of such an arc, it must appear circular or rectilinear in projection; 3d. For this purpose, the plane of the arc must be parallel or perpendicular to the plane of projection; and 4th. This being the case, the axis of the arc will be respectively perpendicular or parallel to the same plane of projection."

We understand by this simply, that, since straight lines and circles are more easily drawn than other lines, it is best to choose an axis of rotation either parallel or perpendicular to the plane of projection; because then the path of a rotating point will be in projection a straight line, or an arc of a circle.

On page 5 we find: "Any magnitude, except a point, which is irreducible to anything more simple, may be regarded as the result of the motion of some simpler magnitude, according to a certain law." Here it is twice implied that a point is a magnitude; while on page 1 it is said that, "Magnitudes are definite portions of *space*, having one, two, or three dimensions." A point is therefore a definite *portion of space*, and has at least one dimension, — probably three, since its length, breadth, and thickness are very likely equal.

On page 104 we find: — "All sections of an oblique cone with a circular base which are parallel to the base, are circles, whose diameter depends on their distance from the vertex. All other sections, whose planes cut all the elements, are ellipses, save the sub-contrary circular section, *EF*, Pl. VIII., Fig. 53, whose plane makes an angle with an element, as *VB*, equal to the angle made by the parallel plane, *CD*, with the opposite element, *VA*."

Here are two omissions. It should have been stated, that the angles with VB and VA are both to be taken on the side towards the vertex. Moreover, there is no intimation that the figure does not represent *any* section of the cone through opposite elements; while it is essential that it should represent the particular section made by a plane drawn through the axis perpendicular to the plane of the base.

In drawing tangents to curves, the author appears to have had in view the method of ROBERVAL. This method is thus stated by its author (*Mém. Acad. des Sciences*, VI. p. 22):—

"*Axiom.* The direction of the motion of a point which describes a curve is a tangent of the curve in each position of this point.

"*General Rule.* By the specific properties of the curve (which are given) examine the different motions which the describing point has at the place where you wish to draw the tangent: compound these motions into one, and draw the line of direction of the compound motion; you will have the tangent of the curve."

PROFESSOR WARREN says (p. 72):—

"The tangent to a curve is therefore the direction in which the generatrix, when at the point of tangency, would move were it to cease to generate a curve, and actually to move in the direction of its tendency for the instant.

"But the law of the generation of a curve is such that the tendency of the generatrix at any instant is the resultant of two component tendencies, while from any common school philosophy we know that such a resultant corresponds to the diagonal of a parallelogram, whose two adjacent sides express the relative values and directions of the component tendencies."

Then follow applications to the conic sections. On pages 284, 385 (the subject being higher curves) we find the method restated thus:—

"The tangent line to the curve, at any point, is the resultant direction of the motion of the generatrix, due to the two component motions of that generatrix.

"The usual case is that neither of these motions is uniform. In this case, the tangent at any point is the resultant of the two tendencies for that instant only. Hence to construct the tangent at any point, draw two straight lines through that point in the directions in which the generatrix tends to move, at that point, and lay off on these lines distances whose ratio shall equal the *ratio* of the *rates* at which the generatrix tends to move at the instant in the given directions. The diagonal of the parallelogram, formed on these distances as two of its sides, will be the required tangent.

"To furnish such momentary ratios as are necessary to the above solutions is one object of the higher calculus."

Now the principle of ROBERVAL's method is acknowledged to be correct; but in applying it there are undoubtedly difficulties in determining what the components are. ROBERVAL himself, MONTUCLA, and even MONGE, in one case, are said by DUHAMEL (*Mém. Sav. Étrang.*, V. 1838, p. 257) to have confounded the component velocities of the generatrix with the projections of the total velocity upon the directions of the components. He shows also, that this confusion will produce no actual error, when the velocities are equal, or when the directions of the motions are at right angles to each other. But when a curve is given by an equation between the distances of its points to two or more foci, these conditions do not in general exist, and the error becomes manifest. To illustrate, suppose we had the equation $ar + br_1 = a$ constant, r and r_1 being the distances from any point of the curve to two fixed points, and a and b any multipliers, and let ϵ and ϵ_1 be the angles made by the tangent to the curve at any point with r and r_1 . Then it is shown by DUHAMEL and by SERRET (*Méthodes en Géométrie*, p. 53, *et seq.*) that

$$\frac{dr}{dr_1} = -\frac{b}{a} = \frac{\cos \epsilon}{\cos \epsilon_1};$$

and, consequently, as will be readily seen by drawing a figure, that if we lay off on r and r_1 any distances having the ratio $dr:dr_1$, and erect perpendiculars at the extremities of these distances, the intersection of the perpendiculars will give the direction of the required tangent. It is obvious that the direction will not be that of the diagonal of a parallelogram constructed on the distances laid off as above, unless either $dr = \pm dr_1$, or the angle between r and r_1 is a right angle.

When, therefore, this method is used, the mode of finding the ratio of the components should be distinctly laid down. We have seen that it is possible to fall into error, even when the Differential Calculus is relied on to furnish the ratios. At any rate, if we have to rely on the Calculus for this essential step, the method ceases to be an independent one, and we had better adopt at once the recognized methods of drawing tangents derived from Analytical Geometry and the Calculus.

Extended as these remarks have become, we have left unnoticed many points that deserve attention. Enough, however, has been said to call attention to the principal features of this work. While we have not hesitated to point out what we consider faults, we trust that we properly appreciate the labors of the author in giving to the English student so extended, and, in many respects, so useful a work.

The Lady's and Gentleman's Diary, or Poetical and Mathematical Almanack for the Year 1861. London: Price one shilling and six pence.

This is the 158th annual number of this interesting and important publication. It is impossible to estimate the influence this unpretending serial has exerted upon the progress of the mathematics, and especially upon the development of mathematical talent in England. Besides 1988 original problems with their solutions, embracing all departments of the science, the mathematical papers thus far published would make several most valuable volumes. In our next we shall give the problems proposed for the next year, in hopes that many of our younger readers will feel disposed to send solutions of them "to the Editor of the Lady's and Gentleman's Diary, Stationer's Hall, London," post-paid, before May 1st, 1861.

Elements of the Differential and Integral Calculus. By WILLIAM SMYTH, A. M., Professor of Mathematics in Bowdoin College. Second Edition. Portland: Published by Sanborn and Carter. 1859.

The luminous statement, given in the Preface, of the spirit and plan of the work, at once arrests the attention; and an examination shows that the author is not only master of his subject, but clearly comprehends the manner of its presentation for the purpose of instruction. We have no hesitation in saying that this is one of the very best text-books upon the subject we have ever seen. The infinitesimal method of LEIBNITZ is adopted as its basis, which the author considers less logical than NEWTON'S Method of Limits or LAGRANGE'S Method of Derived Functions, but better adapted to the purposes of instruction, both on account of its greater simplicity, and the facility with which it is applied to the solution of the ordinary problems demanding the calculus, and especially for all purposes of investigation. Differentials are regarded simply as auxiliary quantities, and those who are familiar with the views of CARNOT (to whom our author acknowledges his indebtedness), as exhibited in his incomparable little work, entitled, *Reflexions sur la Méaphysique du Calcul Infinitésimal*, will at once understand the spirit of the work before us. The arrangement is excellent, as the following brief synopsis will show. I. First Principles. II. Derivation of the auxiliary quantities, or differentials. III. Application to problems in which the differentials can be elim-

inated by the ordinary processes of Algebra. IV. Integral Calculus, in which the auxiliaries, or differentials, are eliminated by reversing the process by which they were obtained. V. Application to problems of quadrature, cubature, rectification; surfaces of solids. VI. Successive differentials, or new auxiliaries. VII. Application of calculus to development of functions. VIII. Maxima and minima. IX. Theory of curves. X. Transcendental functions, their differentiation and integration. XI. Transcendental curves. XII. Processes of integration. XIII. Cycloid. XIV. Quadratures and cubatures continued. XV. Application to Mechanics. XVI. Equilibrium, Centre of Gravity, Hydrodynamics. XVII. Method of Variations. XVIII. Application of Calculus to Astronomy. XIX. Limits, Derived Functions. XX. Miscellaneous examples.

The clearness and simplicity with which the principles are developed, the interesting historical references, and the variety of the problems, all combine to make the work before us an excellent text-book, and as such we commend it to teachers and students, with the assurance that they will not be disappointed.

The methods of NEWTON and LAGRANGE are very properly given at the end of the work, because the student is then prepared to compare them with the method of LEIBNITZ without confusion.

Editorial Items.

WE are happy to lay before our readers the following report; and especially so, because all the Essays submitted are found of "sufficient merit to be worthy of publication." We shall print the Essays in the order named as rapidly as possible, and all in this volume of the Monthly.

REPORT OF THE COMMITTEE ON PRIZE ESSAYS.

The Committee on Prize Essays regret that their Report has been so long delayed. The protracted absence of one of their number and the engagements of the rest prevented an earlier decision. They have received four Essays: On the Conformation of the Earth; On Central Forces; On Projections; and On Spherical Conics. All of these they consider of "sufficient merit to be worthy of publication;" although, as the productions of students, they are not to be too severely criticised. The Committee award the following prizes: To the first Essay, in the order named, forty dollars; to the second, thirty dollars; to the third, twenty dollars; and to the fourth, ten dollars.

WILLIAM FERREL,
J. B. HENCK,
CHAUNCEY WRIGHT.

BOOKS RECEIVED. — General Problems from the Orthographic Projections of Descriptive Geometry, &c., by PROF. S. EDWARD WARREN. (See Notice.) For sale by Brown and Taggard, Boston. — Annual Report of the Board of Regents of the Smithsonian Institution, showing the operations, expenditures, and condition of the Institution for the year 1859. — On the Equation of Differences for an Equation of any Order, and in particular for the Equations of the Orders Two, Three, Four, and Five, by ARTHUR CAYLEY, F. R. S.; also a paper on Recent Terminology in the Mathematics. — Progressive Intellectual, Practical, and Higher Arithmetics; New Elementary and University Algebras; New Plane and Solid Geometry, with Plane and Spherical Trigonometry; Surveying and Navigation, by H. N. ROBINSON, LL. D., published by Ivison, Phinney, & Co., 48 and 50 Walker Street, New York.

THE MATHEMATICAL MONTHLY.

Vol. III . . . MARCH, 1861. . . . No. VI

PRIZE PROBLEMS FOR STUDENTS.

I. Given

$$(1) \quad a^2 - 3x^4 + (2 + 4y)x^2 + (4y + 1 - 2a)x^2 - (2a + 4ay)x = 4y^4,$$

$$(2) \quad x = b + y,$$

to find x and y by quadratics. — Communicated by Prof. E. W. EVANS, Marietta College, Ohio.

II. Describe an isosceles triangle having each of the angles at the base double of the third angle.

III. Which is greater, $2 \tan^{-1}(\sqrt{2} - 1)$, or $3 \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{5}{9}$?
— Communicated by Prof. RICHARD COTTER, Baltimore, Maryland.

IV. In any spherical triangle

$$\frac{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}{\cos S \cos (S-A) \cos (S-B) \cos (S-C)} = \frac{\cos a \cos b \cos c - 1}{\cos A \cos B \cos C + 1},$$

in which $s = \frac{1}{2}(a + b + c)$, $S = \frac{1}{2}(A + B + C)$.*

V. If the earth be completely covered by a sea of small depth, prove that the depth in latitude l is very nearly equal to

$$H(1 - \varepsilon \sin^2 l),$$

where H is the depth at the equator, and ε the ellipticity of the earth.

Solutions of these problems must be received by May 1st, 1861.

* This problem was misprinted in No. XII., Vol. II.

AWARD OF THE PRIZES FOR SOLUTIONS OF PROBLEMS
IN No. II., Vol. III.

The first prize is awarded to GEORGE A. OSBORNE, Jr., Lawrence Scientific School, Cambridge, Mass.

The second prize is awarded to DAVID TROWBRIDGE, Perry City, N. Y.

The third prize is awarded to ASHER B. EVANS, Madison University, Hamilton, N. Y.

The fourth prize, to junior competitors, is not awarded.

PRIZE SOLUTION OF PROBLEM I.

Prove that the surface of a sphere can be completely covered with the surfaces either of 4, or of 8, or of 20 equilateral spherical triangles.

Suppose either a tetrahedron, an octahedron, or an icosahedron to be inscribed in a sphere. Then planes passing through the edges of the polyhedron and the centre of the sphere will divide its surface into either 4, 8, or 20 equilateral triangles. This solution is given by nearly all the competitors.

The solution of Problem II. by the usual formula is obvious.

PRIZE SOLUTION OF PROBLEM III.

By ASHER B. EVANS, Madison University, Hamilton, N. Y.

$$\frac{(x+a+b)^5 + (x+c+d)^5}{(x+a+c)^5 + (x+b+d)^5} = \frac{m}{n}.$$

Find x by quadratics.

Since the sum of the fifth powers of two numbers is divisible by the sum of the numbers, both numerator and denominator of the first member of the given equation is divisible by

$$2x + a + b + c + d.$$

Performing this division, and putting $P = 2x + a + b + c + d$ in the result, we get

$$(1) \quad \frac{P^4 - 5P^2(x+a+b)(x+c+d) + 5(x+a+b)^2(x+c+d)^2}{P^4 - 5P^2(x+a+c)(x+b+d) + 5(x+a+c)^2(x+b+d)^2} = \frac{m}{n}.$$

But $x = \frac{P - a - b - c - d}{2}$, which, substituted in (1), gives, after reduction,

$$(2) \quad \frac{P^4 + 10 \{(a+b)^2 - (c+d)^2\} P^2 + 5 \{(a+b)^2 - (c+d)^2\}}{P^4 + 10 \{(a+c)^2 - (b+d)^2\} P^2 + 5 \{(a+c)^2 - (b+d)^2\}} = \frac{m}{n}.$$

By putting $p = (a+b)^2 - (c+d)^2$, $q = (a+c)^2 - (b+d)^2$, (2) becomes

$$(3) \quad P^4 + \frac{10}{n-m} (np - mq) P^2 = \frac{5}{n-m} (mq - np),$$

or

$$(4) \quad P^4 + 2AP^2 = -A,$$

$$\text{by putting,} \quad \frac{5}{n-m} (np - mq) = A.$$

From (4) we get the four roots,

$$P = \pm (-A \pm \sqrt{(A - \frac{1}{2})^2 - \frac{1}{4}})^{\frac{1}{2}}.$$

These values of P , substituted in $x = \frac{P - a - b - c - d}{2}$, give four of the values of x , and the fifth is given by putting the divisor $2x + a + b + c + d = 0$.

$$\text{Therefore} \quad x = -\frac{a+b+c+d}{2}$$

is the fifth root.

W. D. MERSHON'S solution is similar to the above.

PRIZE SOLUTION OF PROBLEM IV.

By DAVID TROWBRIDGE, Perry City, Schuyler County, N. Y.

A fraudulent tradesman contrives to employ his *false* balance both in buying and selling a certain article, thereby gaining at the rate of 11 per cent more on his outlay than he would gain if the balance were *true*. If however the scale-pans, in which the article is weighed when bought and sold respectively, were interchanged, he would neither gain nor lose by the article. Find the legitimate gain per cent on the article.

Let x denote the fraction of gain or loss in weight by using the false balance, and r the legitimate gain per cent. Every actual unit

of weight weighs $1+x$ when sold, and $1-x$ when bought; he sells each unit for $(1+r)(1+x)$, and buys it for $(1-x)$. His gain is therefore $(1+r)(1+x) - (1-x)$, and the gain per cent is

$$(1) \quad \frac{(1+r)(1+x) - (1-x)}{1-x} = r + \frac{11}{100},$$

by using the false balance. By interchanging the scale-pans he would buy an actual unit of weight for $1+x$, and sell it for $(1+r)(1-x)$; therefore

$$(2) \quad (1+r)(1-x) - (1+x) = 0,$$

since by the question he neither gains nor loses by the transaction.

Equations (1) and (2) give

$$(3) \quad (1+r) \frac{(1+x)}{(1-x)} = 1 + r + \frac{11}{100},$$

$$(4) \quad \frac{1+x}{1-x} = 1 + r.$$

Combining (3) and (4) we get

$$(1+r)^2 - (1+r) = \frac{11}{100}; \quad \therefore r = \frac{1}{10},$$

or 10 per cent is the legitimate gain.

PRIZE SOLUTION OF PROBLEM V.

By GEORGE A. OSBORNE, Jr., Lawrence Scientific School.

Show that in the parabola the half sum of the radii vectores drawn to the extremities of any arc whatever, is equal to the radius vector drawn to the summit of the diameter passing through the centre of the chord and parallel to the axis, plus the part of this diameter intercepted between the arc and the chord.

Let CD be any arc of the parabola, and F the focus. From C and D , the extremities of the chord CD , and also from its middle point, N , drop perpendiculars, DO , NP , and CQ , upon the directrix AB ; and let M be the point in which NP meets the curve. Draw the radii FC , FM , and FD ; then by the known property, that any

point in the curve is equally distant from the focus and directrix, we have

$$\begin{aligned} PN &= \frac{1}{2}(OD + CQ) = \frac{1}{2}(DF + CF) \\ &= PM + MN = MF + MN; \end{aligned}$$

$$\therefore MF + MN = \frac{1}{2}(DF + CF).$$

Several of the competitors gave substantially this solution.

Mr. G. B. HICKS gave the following solution:—

Let the abscissas of the extremities of any arc be designated by x_1 and x_2 , and the radii vectores drawn to these extremities by r_1 and r_2 .

Then calling the equation to the curve $y^2 = 4mx$, we have, from the fundamental property of the parabola,

$$r_1 = x_1 + m, \quad r_2 = x_2 + m,$$

$$\therefore \frac{1}{2}(r_1 + r_2) = \frac{1}{2}(x_1 + x_2) + m.$$

Again, let the radius vector drawn to the summit of the diameter passing through the centre of the chord and parallel to the axis be denoted by r_3 , and the abscissa of the summit by x_3 .

And let h equal the part of the diameter intercepted between the arc and the chord. We have

$$r_3 = x_3 + m.$$

And since the abscissa of the centre of the chord is $\frac{1}{2}(x_1 + x_2)$,

$$h = \frac{1}{2}(x_1 + x_2) - x_3;$$

$$\therefore r_3 + h = \frac{1}{2}(x_1 + x_2) + m = \frac{1}{2}(r_1 + r_2).*$$

* One member of the Committee for awarding these Prizes has resigned. We shall hereafter attend to the duty, as it is often inconvenient for the Committee to report in time for the press. Besides, we have no right to demand the continuance of a serious labor so long and so ably performed. — ED.

FIRST PRIZE ESSAY.—ON THE CONFORMATION OF THE EARTH.

By G. W. HILL, Nyack, N. Y.

1. ALL the particles which compose the mass of the earth are animated by the attraction of gravitation. The law of this force is, that the attraction of any atom for a spherical surface of material points, described about it as a centre, is constant. Hence, if the attraction of an atom for a material point be represented by A , and r be the radius of the spherical surface, and N the number of material points in a unit of surface, the attraction of the central atom for the spherical surface is $4\pi Nr^2 A = \text{a constant} = -4\pi NM$. Whence $A = -\frac{M}{r^2}$; that is, the attraction varies inversely as the distance squared. The constant M is called the mass of the attracting atom. We have given A a negative sign because it represents a force tending to decrease the line r .

2. Making $A = \frac{dV}{dr}$, then $V = \frac{M}{r}$. V is called the *potential function*, and has this property: that if the partial differential of it be taken with respect to any of the three spaceal co-ordinates of which it is necessarily a function, the result will be the partial force in the direction of that co-ordinal axis.

3. If the attraction of a single atom give $V = \frac{M}{D}$, denoting the distance by D , the attraction of an indefinite number or assemblage of atoms will give $V = S \cdot \frac{M}{D}$. If ξ, ψ, φ represent any three lines at right angles with each other, then $\frac{dV}{d\xi}, \frac{dV}{d\psi}, \frac{dV}{d\varphi}$ are the forces acting in each of these three directions respectively. In a rectangular system $D = \{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{\frac{1}{2}}$, the accented letters pertain to the attracting atom, the unaccented to the attracted. Consequently,

$$(1) \frac{dV}{dx} = S \cdot -\frac{M}{D^3}(x-x'), \quad \frac{dV}{dy} = S \cdot -\frac{M}{D^3}(y-y'), \quad \frac{dV}{dz} = S \cdot -\frac{M}{D^3}(z-z').$$

4. Differentiating again,

$$(2) \quad \begin{cases} \frac{d^2 V}{dx^2} = S \cdot \frac{M}{D^3} \left\{ \frac{3}{D^2} (x - x')^2 - 1 \right\}, \\ \frac{d^2 V}{dy^2} = S \cdot \frac{M}{D^3} \left\{ \frac{3}{D^2} (y - y')^2 - 1 \right\}, \\ \frac{d^2 V}{dz^2} = S \cdot \frac{M}{D^3} \left\{ \frac{3}{D^2} (z - z')^2 - 1 \right\}. \end{cases}$$

In this differentiation M has been regarded as independent of x, y, z ; but in order to render equations (2) altogether general, the attracting mass must be considered as extending into the point x, y, z . Let ρ be the density of the atom occupying this point, which becomes zero when the attracting mass does not reach the point. This atom may be regarded as spherical, then for it $M = \frac{4\pi\rho D^3}{3}$; substituting this value in equations (1), the results are

$$\frac{dV}{dx} = \frac{4\pi\rho}{3} (x' - x), \quad \frac{dV}{dy} = \frac{4\pi\rho}{3} (y' - y), \quad \frac{dV}{dz} = \frac{4\pi\rho}{3} (z' - z).$$

Hence we must add the term $-\frac{4\pi\rho}{3}$ to the right members of equations (2), and then we can regard D as having always a finite value. By adding these equations, there results

$$(3) \quad \frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} + 4\pi\rho = 0.$$

The integration of this gives V ; ρ is a function of x, y, z ; in the case of solid bodies, as the earth, a limited function.

5. Transform (3) to terms of polar co-ordinates; put

$$\begin{aligned} x &= r \sqrt{1 - \mu^2} \cos \omega, & r &= \sqrt{x^2 + y^2 + z^2}, \\ y &= r \sqrt{1 - \mu^2} \sin \omega, & \mu &= \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \\ z &= r \mu, & \omega &= \tan^{-1} \frac{y}{x}. \end{aligned}$$

Then

$$\begin{aligned} \frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} &= \left\{ \frac{d^2 r^2}{dx^2} + \frac{d^2 r^2}{dy^2} + \frac{d^2 r^2}{dz^2} \right\} \frac{d^2 V}{dr^2} + \left\{ \frac{d^2 r}{dx^2} + \frac{d^2 r}{dy^2} + \frac{d^2 r}{dz^2} \right\} \frac{dV}{dr} \\ &+ \left\{ \frac{d^2 \mu^2}{dx^2} + \frac{d^2 \mu^2}{dy^2} + \frac{d^2 \mu^2}{dz^2} \right\} \frac{d^2 V}{d\mu^2} + \left\{ \frac{d^2 \mu}{dx^2} + \frac{d^2 \mu}{dy^2} + \frac{d^2 \mu}{dz^2} \right\} \frac{dV}{d\mu} \\ &+ \left\{ \frac{d^2 \omega^2}{dx^2} + \frac{d^2 \omega^2}{dy^2} + \frac{d^2 \omega^2}{dz^2} \right\} \frac{d^2 V}{d\omega^2} + \left\{ \frac{d^2 \omega}{dx^2} + \frac{d^2 \omega}{dy^2} + \frac{d^2 \omega}{dz^2} \right\} \frac{dV}{d\omega}, \\ &= r^{-2} \left\{ \frac{d \cdot r^2 \frac{dV}{dr}}{dr} + \frac{d \cdot (1 - \mu^2) \frac{dV}{d\mu}}{d\mu} + \frac{\frac{d^2 V}{d\omega^2}}{1 - \mu^2} \right\}. \end{aligned}$$

Hence (3) becomes

$$(4) \quad \frac{d \cdot r^2 \frac{dV}{dr}}{dr} + \frac{d \cdot (1 - \mu^2) \frac{dV}{d\mu}}{d\mu} + \frac{\frac{d^2 V}{d\omega^2}}{1 - \mu^2} + 4 \pi \rho r^2 = 0.$$

6. To show the application of (4), take the simple case when the surfaces of equal density are concentrically spherical. Placing the origin of co-ordinates at the common centre, ρ becomes a function of r alone, either continuous or discontinuous, as the case demands; and evidently $\frac{dV}{d\mu} = 0$, $\frac{dV}{d\omega} = 0$; therefore (4) becomes

$$\frac{d \cdot r^2 \frac{dV}{dr}}{dr} + 4 \pi \rho r^2 = 0.$$

By integration, $\frac{dV}{dr} = (C - 4 \pi \rho r^2 dr) r^{-2}$.

$4 \pi \rho r^2 dr$, between the limits 0 and r , is equal to the mass contained within the sphere whose radius is r ; denoting this by M , it is clear that $C = 0$, since the expression must agree with that for the attraction of a single atom. Thus $\frac{dV}{dr} = -\frac{M}{r^2}$. Or the principle may be stated: "The force acting on any point, wherever situated, equals the mass of all the particles nearer the nucleal centre than the point attracted, divided by the square of the point's distance from that centre, with the negative sign."

7. The earth revolves uniformly about a constant axis; hence, to remove our problem from dynamics to statics, it is necessary to in-

roduce the force of pressure called the centrifugal force. Making the co-ordinal axis of z coincide with the earth's axis, and T denoting the period of its revolution, the potential of the centrifugal force is

$$V = \frac{2\pi^2}{T^2} (x^2 + y^2) = \frac{2\pi^2}{T^2} r^2 (1 - \mu^2).$$

Since this force animates every particle, include its potential in V , and make V the potential of both gravitating and centrifugal force. It then becomes necessary to add to (4) the term $-\frac{8\pi^2}{T^2} r^2$,

$$(5) \quad \frac{d \cdot r^2 \frac{dV}{dr}}{d\tau} + \frac{d \cdot (1 - \mu^2) \frac{dV}{d\mu}}{d\mu} + \frac{\frac{d^2 V}{d\omega^2}}{1 - \mu^2} + 4\pi \left(\rho - \frac{2\pi}{T^2} \right) r^2 = 0.$$

8. To apply this equation to the solution of the earth's conformation, we must combine it with some condition of equilibrium. From the manner in which the atmosphere and ocean cover the earth, we may conjecture it was once fluid, and in solidifying preserved the form it had taken by the laws of hydrostatics. In passing from the solid earth to the ocean, and from the ocean to the atmosphere, there occur two faults in the continuity of the earth's density; hence ρ is strictly represented by a discontinuous function. But as the mass of the ocean and atmosphere is about $\frac{1}{4000}$ of the whole, its influence may be neglected, and ρ supposed continuous from centre to surface.

9. If p is the pressure, then $dp = \rho dV$, and $V + B = 0$, is the equation to surfaces of level, B having a different value for every surface. Let ρ be a function of p , and thus of V . In (5) make

$$4\pi \left(\rho - \frac{2\pi}{T^2} \right) = f(V).$$

10. The centrifugal force being small compared with gravity, may be regarded as a perturbing force. Supposing at first this force is zero, the particles would arrange themselves symmetrically about a centre,

since there is no reason why they should accumulate in one place rather than in another. Take the origin of co-ordinates at this centre, then $\frac{dV}{d\mu} = 0$, $\frac{dV}{d\omega} = 0$. Thus (5) becomes

$$\frac{d \cdot r^2 \frac{dV}{dr}}{dr} + r^2 f(V) = 0.$$

Consequently V is a function of r alone, and the general equation to surfaces of level, $V + B = 0$, when solved gives $r = \text{a constant}$; these surfaces are, then, concentrically spherical.

11. The term $\frac{2\pi^2}{T^2} r^2 (1 - \mu^2)$, which the centrifugal force introduces into V , and which causes a departure from the spherical form, does not contain ω , and so whatever derangement it may produce, cannot introduce ω into V ; that is, the earth is a solid of revolution. Consequently (5) becomes

$$(6) \quad \frac{d \cdot r^2 \frac{dV}{dr}}{dr} + \frac{d \cdot (1 - \mu^2) \frac{dV}{d\mu}}{d\mu} + r^2 f(V) = 0.$$

12. From the form of this same term it may be concluded that

$$(7) \quad V = Y_0 + Y_1 \mu^2 + Y_2 \mu^4 + \&c. \dots = \Sigma \cdot Y_i \mu^{2i},$$

where Y_i is a function of r alone, and a quantity of the order of the i th power of the centrifugal force. Substitute this expression of V in (6) and put the coefficient of μ^{2i} resulting equal to zero, and let the coefficient of μ^{2i} in $r^2 f(V)$ be U_i , then

$$(8) \quad \frac{d \cdot r^2 \frac{dY_i}{dr}}{dr} + (2i + 1)(2i + 2) Y_{i+1} - 2i(2i + 1) Y_i + U_i = 0.$$

This equation has the inconvenience of introducing Y_{i+1} ; let us therefore assume more generally $V = \Sigma \cdot V_i M_i$, V_i a function of r , and M_i of μ , V_i being of the same order as Y_i . Substituting in (6),

$$\Sigma \cdot \frac{d \cdot r^2 \frac{dV_i}{dr}}{dr} M_i + \Sigma \cdot V_i \frac{d \cdot (1 - \mu^2) \frac{dM_i}{d\mu}}{d\mu} + r^2 f(\Sigma \cdot V_i M_i) = 0,$$

which may be written

$$(9) \quad \Sigma. \frac{d.r^2 \frac{dV_i}{dr}}{dr} M_i + \Sigma. V_i \frac{d.(1-\mu^2) \frac{dM_i}{d\mu}}{d\mu} + \Sigma. T_i M_i = 0.$$

In order that the left-hand member may be arranged in a series of the same form as $\Sigma. V_i M_i$, we must have

$$\frac{d.(1-\mu^2) \frac{dM_i}{d\mu}}{d\mu} = n M_i,$$

in which n is independent of μ . We may determine n from the consideration that V_i being of the same order as Y_i , M_i cannot contain any higher power of μ than μ^{2i} . Making $M_i = \Sigma. k_i \mu^{2i}$, this relation results: $k_{s+1} = \frac{n+2s(2s+1)}{(2s+1)(2s+2)} k_s$. To make this series end at k_i , n must equal $-2i(2i+1)$; and $k_{s+1} = -\frac{(2i-2s)(2i+2s+1)}{(2s+1)(2s+2)} k_s$; hence, putting $k_0 = 1$, which is allowable,

$$(10) \quad M_i = 1 - \frac{2i(2i+1)}{1.2} \mu^2 + \frac{(2i-2)2i(2i+1)(2i+3)}{1.2.3.4} \mu^4 \dots \\ \pm \frac{2 \dots 2i(2i+1) \dots (4i+1)}{1.2.3 \dots 2i} \mu^{2i}.$$

For V_i we have from (9), by rejecting the sign Σ and dividing by M_i , this equation:—

$$(11) \quad \frac{d.r^2 \frac{dV_i}{dr}}{dr} - 2i(2i+1) V_i + T_i = 0.$$

From the expression (10) we easily deduce

$$(12) \quad V_i = \pm \frac{(2i+1)(2i+3) \dots (4i-1)}{1.3 \dots (2i-1)} \left\{ V_i + \frac{(i+1)(4i+1)}{1.(2i+1)} V_{i+1} \right. \\ \left. + \frac{(i+1)(i+2)(4i+1)(4i+3)}{1.2.(2i+1)(2i+3)} V_{i+2} + \&c. \dots \right\}.$$

The reversion of which is

$$(13) \quad V_i = \pm \frac{1.3 \dots (2i-1)}{(2i+1)(2i+3) \dots (4i-1)} \left\{ Y_i + \frac{(i+1)(2i+1)}{1.(4i+3)} Y_{i+1} \right. \\ \left. + \frac{(i+1)(i+2)(2i+1)(2i+3)}{1.2.(4i+3)(4i+5)} Y_{i+2} + \&c. \dots \right\}.$$

The upper sign is to be used when i is even, the lower when odd. From (10) we obtain

$$(14) \quad M_1^2 = \frac{1}{6} - \frac{1}{3} M_1 + \frac{2}{3} M_2.$$

From this and the equation $r^2 f(\Sigma . V_i M_i) = \Sigma . T_i M_i$, pursuing the approximation to quantities of the second order, we get these expressions for T_i ,

$$(15) \quad \begin{cases} r^{-2} T_0 = f(V_0) + \frac{2}{3} V_1^2 f''(V_0) + \&c. \dots \\ r^{-2} T_1 = V_1 f'(V_0) - \frac{2}{3} V_1^2 f''(V_0) + \&c. \dots \\ r^{-2} T_2 = V_2 f'(V_0) + \frac{2}{3} V_1^2 f''(V_0) + \&c. \dots \end{cases}$$

If quantities of the second order are neglected, the two differential equations to be integrated are

$$(16) \quad \begin{cases} \frac{d . r^2 \frac{d V_0}{d r}}{d r} + r^2 f(V_0) = 0, \\ \frac{d . r^2 \frac{d V_1}{d r}}{d r} - 6 V_1 + r^2 V_1 f'(V_0) = 0. \end{cases}$$

To pursue the analysis farther would require a knowledge of the form of $f(V)$.

13. However, when the point r, μ, ω is without the surface of the earth, (11) can be integrated. Supposing the point not to partake of the motion of revolution, the centrifugal force must be neglected, and since $\rho = 0$, generally $T_i = 0$; consequently (11) becomes

$$(17) \quad \frac{d . r^2 \frac{d \check{V}_i}{d r}}{d r} - 2i(2i+1) \check{V}_i = 0.$$

The integral of this is $\check{V}_i = a_i r^{2i} + b_i r^{-2i+1}$, in which $a_i = 0$; otherwise the earth's attraction would be infinitely great at an infinite distance. From the equation $V = \Sigma . V_i M_i$, by giving i successively the values 0, 1, 2, &c., we have, then,

$$(18) \quad \check{V} = b_0 r^{-1} + b_1 r^{-3}(1-3\mu^2) + b_2 r^{-5}(1-10\mu^2+\frac{35}{8}\mu^4) + \&c. \dots$$

We have written V in this case $\bar{V}$. If r, μ, ω is situated in the atmosphere or at the surface, and revolves with the earth, we have only to add to $\bar{V}$ the potential of the centrifugal force $\frac{2\pi^2}{T^2} r^2 (1 - \mu^2)$, and may then call it $\bar{V}$; and for points within write $\hat{V}$.

14. The integration of (11) introduces an indefinite number of constants, of which the superfluous are to be eliminated in the usual way of treating partial differential equations, viz.: by equations of limitation; which in the present question are the equations having places at the limit of the attracting mass. At this limit $\frac{dV}{dr}, \frac{dV}{d\mu}, \frac{dV}{d\omega}$, derived from the integration of (5), must have the same value, whether ρ in that equation equals zero or its value at the surface. Consequently we have

$$(19) \quad \frac{d.(\bar{V}-\bar{V})}{dr} = 0, \quad \frac{d.(\bar{V}-\bar{V})}{d\mu} = 0, \quad \frac{d.(\bar{V}-\bar{V})}{d\omega} = 0.$$

Hydrostatics furnishes two other equations,

$$(20) \quad \hat{V} + B = 0, \quad \bar{V} + B' = 0.$$

These five are all equations to the same surface, of which it must be noticed that they are not all independent, but any one is a consequence of the other four. They will give all the conditions needed to eliminate the superfluous arbitrary constants. The last of (19) is, since V does not involve ω , an identical equation; select then the first of (19) and the two of (20) for the purpose of elimination. It is evident from $V = \sum V_i M_i$, that each one of these equations to the surface can be transformed to this shape: $r = \sum c_i M_i$, c_i a constant and of the i th order with respect to the centrifugal force. By substituting this value of r in equations (19) and (20), and equating the coefficients of M_i generally to zero, and rejecting the two involving B and B' , we have, pursuing the approximation to the i th order, $3i + 1$ equations. The constants they involve are the con-

stants resulting from the integration of (11), $2i + 2$ in number, and the constants c_i , $i + 1$ in number, in all $3i + 3$. Thus two constants are left indeterminate, which is as it should be, since the units of length and density are arbitrary. We may assume, then, $c_0 = 1$; making this substitution in $\bar{V} + B = 0$, using accents to denote differentiation and putting r afterwards equal to unity, and taking account of quantities of the second order, we have

$$(21) \quad \begin{cases} V_0' c_1 + V_1 - \frac{2}{3} c_1 \{ V_0'' c_1 + 2 V_1' \} = 0. \\ V_0' c_2 + V_2 + \frac{2}{3} c_1 \{ V_0'' c_1 + 2 V_1' \} = 0. \end{cases}$$

Treating $\bar{V} + B' = 0$ in the same way, two similar equations result, which can be evidently derived from (21) by making

$$V_0 = b_0 r^{-1} + \frac{4 \pi^2}{3 T^2} r^2, \quad V_1 = b_1 r^{-3} + \frac{2 \pi^2}{3 T^2} r^2, \quad \text{and} \quad V_2 = b_2 r^{-5},$$

$$(22) \quad \begin{cases} \left\{ -b_0 + \frac{8 \pi^2}{3 T^2} \right\} c_1 + b_1 + \frac{2 \pi^2}{3 T^2} - \frac{2}{3} c_1 \left\{ 2 b_0 c_1 - 6 b_1 + \frac{8 \pi^2}{3 T^2} \right\} = 0, \\ -b_0 c_2 + b_2 + \frac{2}{3} c_1 \left\{ 2 b_0 c_1 - 6 b_1 + \frac{8 \pi^2}{3 T^2} \right\} = 0. \end{cases}$$

From the equation $\frac{d \cdot (\bar{V} - V)}{dr} = 0$, also we have

$$(23) \quad \begin{cases} V_0' + \frac{2}{3} c_1 \{ V_0''' c_1 + 2 V_1'' \} = -b_0 + \frac{8 \pi^2}{3 T^2} + \frac{2}{3} c_1 \left\{ -6 b_0 c_1 + 24 b_1 + \frac{8 \pi^2}{3 T^2} \right\}, \\ V_0'' c_1 + V_1' - \frac{2}{3} c_1 \{ V_0''' c_1 + 2 V_1'' \} = \left(2 b_0 + \frac{8 \pi^2}{3 T^2} \right) c_1 - 3 b_1 + \frac{4 \pi^2}{3 T^2} \\ \quad - \frac{2}{3} c_1 \left\{ -6 b_0 c_1 + 24 b_1 + \frac{8 \pi^2}{3 T^2} \right\}, \\ V_0'' c_2 + V_2' + \frac{2}{3} c_1 \{ V_0''' c_1 + 2 V_1'' \} = 2 b_0 c_2 - 5 b_2 \\ \quad + \frac{2}{3} c_1 \left\{ -6 b_0 c_1 + 24 b_1 + \frac{8 \pi^2}{3 T^2} \right\}. \end{cases}$$

Equations (22) and (23) can be reduced to the following simpler forms:—

$$(24) \begin{cases} b_0 = -\{1 + \frac{3}{2}q + \frac{1}{6}q^2 - \frac{3}{2}q c_1 + \frac{3}{2}c_1^2\} \{V_0' + \frac{3}{2}c_1 [V_0''' c_1 + 2V_1'']\}, \\ b_1 = \{ (1 - \frac{3}{2}q) c_1 - \frac{3}{2}c_1^2 - \frac{q}{6} (1 - \frac{q}{2}) b_0, \\ b_2 = \{c_2 - \frac{3}{2}q c_1 + \frac{3}{2}c_1^2\} b_0, \end{cases}$$

$$(25) \begin{cases} V_0'' c_1 + V_1' - \frac{3}{2}c_1 \{V_0''' c_1 + 2V_1''\} - \{ (1 - \frac{3}{2}q) c_1 + \frac{1}{2}c_1^2 - \frac{3}{2}q (1 + \frac{q}{6}) \} V_0' = 0 \\ V_0'' c_2 + V_2' + \frac{3}{2}c_1 \{V_0''' c_1 + 2V_1''\} - \{3c_2 - \frac{3}{2}q c_1 + \frac{3}{2}c_1^2\} V_0' = 0, \end{cases}$$

if we make $\frac{4\pi^2}{T^2} = q \{b_0(1 - 3c_1) + 3b_1\}$.

15. In this article we shall neglect quantities of the order of q^2 . Let $\frac{w}{3}$ denote the quantity obtained by dividing the second member of the second equation of (23) by the second member of the first equation, then

$$w = 3 \cdot \frac{-3b_1 + \frac{4\pi^2}{3T^2} + 2b_0c_1}{b_0} = 6c_1 + q - 9\frac{b_1}{b_0};$$

from the second equation of (24), $3c_1 = \frac{3b_1}{b_0} + \frac{q}{2} = e$, the approximate value of the earth's compression, as is clear from the equation $r = 1 + c_1(1 - 3\mu^2) + \&c...$ Hence by addition

$$(26) \quad e + w = 6c_1 + \frac{3}{2}q - 6\frac{b_1}{b_0} = \frac{5}{2}q.$$

The relation enunciated by this equation is known as CLAIRAUT'S Theorem.

16. The equation of the earth's surface is

$$(27) \begin{aligned} r &= 1 + c_1(1 - 3\mu^2) + c_2(1 - 10\mu^2 + \frac{3}{2}\mu^4) + \&c... \\ &= (1 + c_1 + c_2) \{1 - [3c_1(1 - c_1) + 10c_2]\mu^2 + \frac{3}{2}c_2\mu^4 + \&c...\} \\ &= a \{1 + e_1 \sin^2 \theta + e_2 \sin^4 \theta + \&c... \}; \end{aligned}$$

in which θ is the geocentric latitude. If e represent the compression, or part of its own length by which the equatorial radius exceeds the polar,

$$(28) \quad e = -e_1 - e_2 = 3c_1(1 - c_1) - \frac{5}{2}c_2.$$

17. Denoting the normal or astronomical latitude by θ' , we have

$$(29) \quad \theta' = \theta - \tan^{-1} \frac{dr}{r d\theta} = \theta - \left\{ e_1 + e_2 - \frac{e_1^2}{2} \right\} \sin 2\theta - \frac{2e_2 - e_1^2}{4} \sin 4\theta,$$

the reversion of which is

$$(30) \quad \theta = \theta' + \left\{ e_1 + e_2 - \frac{e_1^2}{2} \right\} \sin 2\theta' + \frac{2e_2 + 3e_1^2}{4} \sin 4\theta'.$$

18. A line geodetically measured on the earth's surface is clearly the shortest possible; hence, if ds is the element of the curve, by the principles of the Calculus of Variations, $d \cdot f ds + \lambda \delta V = 0$, λ being the indeterminate multiplier of δV . Also,

$$ds^2 = dr^2 + \frac{r^2 d\mu^2}{1-\mu^2} + r^2(1-\mu^2)d\omega^2.$$

The coefficients of δr , $\delta \mu$, $\delta \omega$ each equal zero; retaining only that of $\delta \omega$, as sufficient for our purpose, we have

$$d \cdot \left\{ r^2(1-\mu^2) \frac{d\omega}{ds} \right\} = 0, \quad \text{or} \quad r^2(1-\mu^2) \frac{d\omega}{ds} = ah.$$

That is, the sine of the angle made by this curve with the meridians varies inversely as the distance from the earth's axis. Hence

$$\frac{d\omega}{d\mu} = \frac{ah \sqrt{1 + \left(\frac{dr}{r d\mu}\right)^2 (1-\mu^2)}}{(1-\mu^2) \sqrt{r^2(1-\mu^2) - a^2 h^2}} \quad \text{and} \quad \frac{ds}{d\mu} = \frac{r^2 \sqrt{1 + \left(\frac{dr}{r d\mu}\right)^2 (1-\mu^2)}}{\sqrt{r^2(1-\mu^2) - a^2 h^2}}.$$

Taking account only of the first order with respect to q ,

$$d\omega = \frac{h d\mu}{(1-\mu^2) \sqrt{(1-h^2-\mu^2)}} - \frac{h e_1 \mu^2 d\mu}{(1-h^2-\mu^2)^{\frac{3}{2}}}.$$

$$(31) \quad \omega = C - \sin^{-1} \frac{1 - \frac{1+h^2}{1-h^2} \mu^2}{1-\mu^2} + h e_1 \left\{ \sin^{-1} \frac{\mu}{\sqrt{(1-h^2)}} - \frac{\mu}{\sqrt{(1-h^2-\mu^2)}} \right\}.$$

This, then, is the equation to the curve; if ω , ω_{∞} are the extreme values of ω , and μ , μ_{∞} those of μ , h can be found from the expression

for $\omega'' - \omega$. If ε is the angle made by the curve with the meridian at the commencement, then $h = \cos \theta$, $\sin \varepsilon (1 + e_1 \sin^2 \theta)$, and as an approximate value of ε , we have

$$\cot \varepsilon = \frac{\tan \theta'' \cos \theta'}{\sin (\omega'' - \omega')} - \sin \theta, \cot (\omega'' - \omega).$$

For the length of the curve,

$$ds = a \left\{ \frac{1 + e_1 \mu^2}{\sqrt{(1 - h^2 - \mu^2)}} - \frac{h^2 e_1 \mu^2}{(1 - h^2 - \mu^2)^{\frac{3}{2}}} \right\} d\mu,$$

$$(32) \quad s = a \left\{ \left[1 + (1 + h^2) \frac{e_1}{2} \right] \sin^{-1} \frac{\mu}{\sqrt{(1 - h^2)}} - \frac{e_1}{2} \frac{(1 + h^2 - \mu^2) \mu}{\sqrt{(1 - h^2 - \mu^2)}} \right\}.$$

If $h = 0$, this expression gives the length of any arc of a meridian, but in this case,

$$ds^2 = \sqrt{(dr^2 + r^2 d\theta^2)} = r \left\{ 1 + \frac{e_1^2}{2} \sin^2 2\theta \right\} d\theta,$$

or

$$ds = a \left\{ 1 + e_1 \sin^2 \theta + e_2 \sin^4 \theta + \frac{e_1^2}{2} \sin^2 2\theta \right\} d\theta,$$

$$(33) \quad s = a \left\{ \left(1 + \frac{e_1}{2} + \frac{e_1^2}{4} + \frac{3}{8} e_2 \right) \theta - \frac{e_1 + e_2}{4} \sin 2\theta + \frac{2e_1^2 - e_2}{32} \sin 4\theta \right\}.$$

19. All areas on the earth's surface bounded by lines, whose equation is (31), can be divided into a finite number of parts, each contained by an arc of a meridian, an arc of a parallel of latitude, and a line whose equation is (31). Let A denote the area of this, then ds , being the element of the meridian, $A = \int \omega r \cos \theta ds$, or, neglecting quantities of the second order, $= \int \omega r^2 d\mu = ar \mu \omega - afr \mu d\omega$, finally putting for $d\omega$ its value, and integrating: —

$$(34) \quad A = ar \mu \omega + \frac{a^2 h}{2} \left\{ \frac{1 + e_1}{h} \tan^{-1} \frac{\sqrt{(1 - h^2 - \mu^2)}}{h} + \frac{(1 - h^2) e_1}{\sqrt{(1 - h^2 - \mu^2)}} \right\}.$$

20. Let F denote the force of gravity at the surface, then

$$F = \frac{dV}{dr} \sqrt{1 + \left(\frac{dr}{r d\theta} \right)^2} = \frac{dV}{dr} \{ 1 + 18 e_1^2 \mu^2 (1 - \mu^2) \}.$$

If A_0, A_1, A_2 are the numerical values of the members of equations (23), then

$$F = (A_0 + A_1 + A_2) \left\{ 1 - \frac{3A_1 + 10A_2 - 18V_0'c_1^2}{A_0 + A_1} \mu^2 + \frac{3A_2 - 18V_0'c_1^2}{A_0} \mu^4 \right\}.$$

$$(35) = F_0 \{ 1 + w_1 \mu^2 + w_2 \mu^4 + \&c. \dots \}.$$

21. Thus far the general theory of the subject. We shall now assume some particular law of density. Suppose that the matter of which the earth is composed is compressible inversely as its density. This gives $d\rho = \frac{m^2}{4\pi} \frac{dp}{\rho}$, m being a constant. Substituting for dp its value, ρdV , and integrating $\rho = \frac{m^2}{4\pi} V$, no constant is added, because it may be supposed contained in V . Then

$$f(V) = m^2 V - \frac{8\pi^2}{T^2},$$

and from (15) generally $T_i = m^2 r^2 V_i$, but

$$T_0 = m^2 r^2 \left(V_0 - \frac{8\pi^2}{m^2 T^2} \right).$$

Thus (11) becomes

$$(36) \quad \frac{d \cdot r^2 \frac{dV_i}{dr}}{dr} - \{ 2i(2i+1) - m^2 r^2 \} V_i = 0.$$

In integrating, the part of V_i involving negative powers of r may be neglected, since it belongs to $\check{V}$. If a_i is an arbitrary constant, and f^{2i} denotes the operation $\frac{d}{dr} \cdot \frac{1}{r}$ performed $2i$ times, the integral of (36) is $V_i = a_i r^{2i-1} f^{2i}(\sin mr)$. The integral may also be obtained thus: put (36) under this form, —

$$(37) \quad \frac{d^2 \cdot r V_i}{dr^2} + \left\{ m^2 - \frac{2i(2i+1)}{r^2} \right\} r V_i = 0.$$

Assume $r V_i = P \sin mr + P' \cos mr$; which by substitution gives

$$\frac{d^2 \cdot (P \pm P')}{dr^2} \pm 2m \frac{d \cdot (P \mp P')}{dr} - \frac{2i(2i+1)}{r^2} (P \pm P') = 0.$$

Make $P \pm P' = \beta_0 \pm \beta_1 r^{-1} + \beta_2 r^{-2} \pm \beta_3 r^{-3} + \&c. \dots$; then this equation results,

$$\pm 2m(n+1)\beta_{n+1} + \{n(n+1) - 2i(2i+1)\}\beta_n = 0;$$

whence
$$\beta_{n+1} = \mp \frac{(n-2i)(n+2i+1)}{(n+1)2m} \beta_n,$$

the upper sign being taken when n is even, the lower when odd. Then making $\beta_0 = \pm m^{2i} a_i$, in order to agree with the expression $V_i = a_i r^{2i-1} f^{2i}(\sin mr)$, we have

$$(38) \quad \beta_n = \pm \frac{(2i-n+1) \dots (2i+n)}{1 \cdot 2 \dots n \cdot 2^n} m^{2i-n} a_i;$$

the upper sign having place when $2i-n$ is of the forms $4\nu+2$, $4\nu+3$, the lower when it is of the forms 4ν , $4\nu+1$.

$$(39) \quad V = \frac{8\pi^2}{m^2 T^2} + a_0 r^{-1} f^0(\sin mr) + a_1 r f^2(\sin mr) \{1 - 3\mu^2\} \\ + a_2 r^3 f^4(\sin mr) \{1 - 10\mu^2 + \frac{35}{8}\mu^4\} + \&c. \dots;$$

or, expanding $f^{2i}(\sin mr)$ by using (38),

$$(40) \quad V = \frac{8\pi^2}{m^2 T^2} + a_0 \frac{\sin mr}{r} + a_1 \left\{ \left(\frac{3}{r^2} - m^2 \right) \frac{\sin mr}{r} - \frac{3m \cos mr}{r} \right\} \{1 - 3\mu^2\} \\ + a_2 \left\{ \left(\frac{105}{r^4} - \frac{45m^2}{r^2} + m^4 \right) \frac{\sin mr}{r} - \left(\frac{105m}{r^3} - \frac{10m^3}{r} \right) \frac{\cos mr}{r} \right\} \\ \{1 - 10\mu^2 + \frac{35}{8}\mu^4\} + \&c. \dots$$

22. Since V contains a constant, the term $\frac{8\pi^2}{m^2 T^2}$ may be neglected, except in finding the value of the density. Moreover, for simplicity, let $a_0 = 1$, and $\frac{V'_0}{m^2 V_0} = -H$; then from (36) and (40) we derive

$$(41) \quad V_0'' = (2H-1)m^2 V_0, \\ V_0''' = (2+(m^2-6)H)m^2 V_0, \\ V_1 = a_1(3H-1)m^2 V_0,$$

$$\begin{aligned} V_1' &= a_1 (3 + (m^2 - 9) H) m^2 V_0, \\ V_1'' &= a_1 (m^2 - 12 - (5 m^2 - 36) H) m^2 V_0, \\ V_2 &= a_2 (m^2 - 35 - (10 m^2 - 105) H) m^2 V_0, \\ V_2' &= a_2 (-10 m^2 + 175 - (m^4 - 65 m^2 + 525) H) m^2 V_0. \end{aligned}$$

By substituting these expressions in (21) and (25), neglecting quantities of the second order, they become, after removing the factor $m^2 V_0$,

$$(3 H - 1) a_1 - H c_1 = 0,$$

and

$$\{3 + (m^2 - 9) H\} a_1 + (3 H - 1) c_1 - \frac{5}{8} q H = 0;$$

whence

$$(42) \quad a_1 = \frac{5}{8} q \frac{H^2}{m^2 H^2 - (3 H - 1)}, \quad c_1 = \frac{5}{8} q \frac{H(3 H - 1)}{m^2 H^2 - (3 H - 1)}.$$

23. Represent the volume of the earth by v , its superficial density by R , its mean density by R' . Then true to quantities of the first order, $v = \frac{4\pi}{3}$; and if (V) denote the value of V at the surface $R = \frac{m^2}{4\pi} (V)$. Since b_0 is the mass of the earth, $R' = \frac{b_0}{v} = \frac{3 b_0}{4\pi}$. Hence $\frac{m^2(V)}{b_0} = \frac{3 R}{R'}$, or, putting for b_0 its value from (24),

$$\frac{V_0'}{m^2(V)} = - \frac{R'}{3 R (1 + \frac{5}{8} q)}.$$

If in the expressions for V we make $3 \mu^2 = 1$, and consequently $r = 1$,

$$(V) = \frac{8 \pi^2}{m^2 T^2} + V_0 = - \frac{2 q}{m^2} V_0', \quad \text{then} \quad \frac{H}{1 + 2 q H} = \frac{R'}{3 R (1 + \frac{5}{8} q)}, \quad \text{or}$$

$$(43) \quad H = \frac{R'}{3 R - 2 q (R' - R)}.$$

To find m we have, since $H = - \frac{V_0'}{m^2 V_0}$,

$$(44) \quad \frac{1}{m} \left(\frac{1}{m} - \cot m \right) = H.$$

24. In order to test the preceding theory by numerical calculation, we adopt the following values for q , R , R' , the best we can find : —

$$q = \frac{1}{2\frac{1}{3}}, \quad R = 2.55, \quad R' = 5.67.$$

We shall mark with an accent the numbers of the formulas from which the numerical values of the following quantities are obtained : —

$$\begin{aligned} (43)' \quad H &= 0.739083, \\ (44)' \quad \begin{cases} m &= 146^\circ 12' 19''.8, \\ &= 2.551765, \end{cases} \\ (42)' \quad \begin{cases} a_1 &= 0.00067556, \\ c_1 &= 0.0011126, \end{cases} \\ (41)' \quad V_0'' &= 0.478166 m^2 V_0, \end{aligned} \quad (41)' \quad \begin{cases} V_0''' = 2.378 m^2 V_0, \\ V_1 = 1.217249 m^2 V_0 a_1, \\ V_1' = 1.160794 m^2 V_0 a_1, \\ V_1'' = -2.944 m^2 V_0 a_1, \\ V_2 = 0.990 m^2 V_0 a_2, \\ V_2' = 3.345 m^2 V_0 a_2. \end{cases}$$

To obtain the values of a_1 , c_1 true to quantities of the order of q^2 , by substituting the preceding in (21) and (25), we have

$$\begin{aligned} (21)' \quad 0.739083 c_1 - 1.217249 a_1 + 0.000000666 &= 0, \\ (25)' \quad 1.217249 c_1 + 1.160794 a_1 - 0.002141513 &= 0. \\ (21)' \quad 0.739 c_2 - 0.990 a_2 - 0.000000900 &= 0, \\ (25)' \quad 2.695 c_2 + 3.345 a_2 - 0.000005243 &= 0. \end{aligned}$$

$$\begin{aligned} a_1 &= 0.000676848, & c_1 &= 0.001113848, \\ a_2 &= 0.000000283, & c_2 &= 0.000001595. \end{aligned}$$

$$\begin{aligned} (18)' \& (24)' \quad \tilde{V} &= b_0 \{ r^{-1} + 0.0005430 r^{-3} (1 - 3 \mu^2) \\ &\quad + 0.0000010 r^{-5} (1 - 10 \mu^2 + \frac{25}{3} \mu^4) \}. \\ (27)' \quad r &= a \{ 1 - 0.003353781 \sin^2 \theta + 0.000018608 \sin^4 \theta \}. \\ (28)' \quad e &= 0.003335173 = \frac{1}{299.834}. \\ (29)' \quad \theta' &= \theta + 679''.09 \sin 2 \theta - 1''.34 \sin^4 \theta. \\ (30)' \quad \theta &= \theta' - 679''.09 \sin 2 \theta' + 3''.66 \sin^4 \theta'. \\ (33)' \quad s &= a \{ 0.9983329 \theta + 0.0008338 \sin 2 \theta \\ &\quad + 0.0000001 \sin 4 \theta \}. \\ (35)' \quad F &= F_0 \{ 1 + 0.0054093 \sin^2 \theta - 0.0000539 \sin^4 \theta \}. \end{aligned}$$

The following table contains the values of ρ and of e , the compression of the surfaces of level, for every tenth of the equatorial radius, calculated from the equations

$$\rho \Rightarrow \frac{m^2}{4\pi} V = \frac{R \sin mr}{r \sin m}, \quad \text{and} \quad e = -\frac{3V_1}{V_0'} = 3a_1 \left\{ \frac{3}{r} - \frac{m}{\frac{1}{mr} - \cot mr} \right\}.$$

$\frac{r}{a}$	ρ	e	$\frac{r}{a}$	ρ	e
0.0	11.699	0000	0.6	7.635	1-586th
0.1	11.572	1-3675th	0.7	6.398	1-488th
0.2	11.197	1-1872th	0.8	5.108	1-412th
0.3	10.589	1-1240th	0.9	3.810	1-349th
0.4	9.770	1-915th	1.0	2.550	1-300th
0.5	8.773	1-720th			



NOTES AND QUERIES.

I. A LOGICAL OUTLINE OF ARITHMETIC. — One great error in our systems of elementary instruction is the lack of logical method. With the desire of lending a word toward the correction of this error, we present the following logical outline of Arithmetic.

All numerical ideas begin with the unit. It is the origin, the basis of Arithmetic. The unit can be increased and divided, hence arise Numbers and Fractions. Every number is a *synthesis*, every fraction the result of an *analysis*. When by synthesis we have obtained a number, by analysis we may pass to a smaller number; hence numbers can be increased and diminished, and these are the only operations to which they may be subjected. All other processes are merely modifications of, or are based upon, these fundamental ones.

To determine *when* and *how* to increase, and *when* and *how* to diminish, we employ a process of reasoning called Comparison. This

reasoning process also gives rise to several particular arithmetical processes. Arithmetic of whole numbers — and the same obtains with fractions — is therefore reduced to the two general processes of Synthesis and Analysis, these to be controlled by the logic of Comparison. Let us glance briefly at each.

A general synthesis is Addition. Multiplication is a special case of addition, in which the numbers to be added are equal, — the sum in this case being distinguished by the term *product*. Composite numbers, formed by the synthesis of factors; Multiples formed by the synthesis of *particular* factors, and Involution, requiring a product of *equal* factors, are all included under Multiplication.

A general analysis is called Subtraction. Division is a *special* analysis, in which equal numbers are successively subtracted, with the additional object of ascertaining how many times such subtractions may be performed. Factoring is a special case of division, in which many or all of the divisors of a member are required; Evolution, a special case of factoring, in which *one* of several *equal* factors is required; and Common Divisor, a case of factoring in which some common factor of several numbers is required.

In comparing numbers we observe we may pass from one to another of different species under the same genus, and thus we have Reduction. By Comparison we obtain Ratio, arithmetical and geometrical, from the first of which, by a farther comparison, arises Arithmetical Progression; and from the second we obtain, in a similar manner, Proportion and Geometrical Progression.

Thus we derive a complete outline of the science of numbers. The rest of Arithmetic consists of the solution of problems, either real or theoretic, and may be included under the head of Applications of Numbers. Arithmetic is therefore pure and applied, or abstract and concrete. The following is our logical outline of abstract Arithmetic: —

SYNTHESIS.	Addition.	{ Multiplication.	{	Forming Composite Numbers.
				Involution.
ANALYSIS.	Subtraction.	{ Division.	{	Multiples.
				Factoring.
COMPARISON.	{	{	{	Evolution.
				Common Divisor.
				Reduction.
				Ratio.
				Proportion.
				Progressions.

It is believed that the above view of Arithmetic must tend to simplify the subject, and that much clearer notions of the science will be obtained when these philosophical relations are understood. — EDWARD BROOKS, Prof. of Math., State Normal School, Millersville, Pa.

II. *Note on finding the expressions for the sines and cosines of the sum and difference of two angles in terms of the sines and cosines of the angles themselves, by using the equations for passing from a system of rectangular co-ordinates to another system also rectangular.*

These equations are as follows : —

$$(1) \quad y = x' \sin a + y' \cos a,$$

$$(2) \quad x = x' \cos a - y' \sin a.$$

But $y = \sin (\alpha + \beta),$ and $x = \cos (\alpha + \beta).$

Also $y' = \sin \beta,$ and $x' = \cos \beta.$

Substituting these values in (1) and (2), we have

$$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha,$$

and $\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.$

If the point P falls between the axes of x , then y' is negative, and the equations become

$$\begin{aligned} \sin (\alpha - \beta) &= \sin \alpha \cos \beta - \sin \beta \cos \alpha, \\ \text{and} \quad \cos (\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta. \end{aligned}$$

— M. C. STEVENS, Professor of Mathematics, Haverford College, Pa.

III. The following similarity in the form of equations may not be uninteresting to some readers of the Monthly.

$$\frac{x}{a} + \frac{y}{b} = 1 \text{ is the equation of the straight line,}$$

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1 \text{ is the equation of ellipse or hyperbola,}$$

according as we take the upper or lower sign, the origin being the centre.

$$\frac{xx'}{a^2} \pm \frac{yy'}{b^2} = 1 \text{ is the equation to the tangent line to the ellipse or hyperbola.}$$

In these equations, a is the intercept on the axis of x , and b is the intercept on the axis of y .

In the case of the hyperbola, the intercept on the axis of y is imaginary.

These forms, by their similarity, are easily remembered. I take them from SALMON'S Conic Sections. — M. C. S.

IV. *Notes on Geometry.* — We think that nothing ought to be admitted in Geometry as self-evident which admits of proof. The only basis of geometrical reasoning ought to be the definitions, and those intuitive truths which can be reduced to no simpler elements, and which it is impossible to deny.

Now it can be *proved* that all right angles are equal, and LEGENDRE has given the proof satisfactorily; why, then, is this truth taken for granted in most of our Geometries? It does not follow that because it is sufficiently obvious to some minds, it is so to all.

Again, in order that any assumed measure of the inclination of two planes may be a just measure, it is necessary that the measure shall be equal for equal inclinations, that it shall vary as the inclinations vary, and that it shall be the same at whatever point of the intersection of the two planes it may be applied; and yet, with all these conditions to be satisfied, it is taken for granted that the angle contained between two perpendiculars, one in each plane and perpendicular to the common intersection at the same point, is a just measure of the inclination of the planes. It is, to say the least, quite as self-evident that an angle is measured by the arc intercepted between its sides and described from its vertex as a centre. Then why demonstrate this last?

It is more excusable to assume, that only one line can be drawn through a given line parallel to a given line, because it was very difficult to prove this proposition in an elementary way, and very difficult to establish the theory of parallel lines without it.

The following method seems elementary enough and entirely satisfactory. Let ab and de be two parallels, and ac a line making any angle with ab . Now however small the angle bac , it may be made, by being repeated a definite number of times around a , to fill up the whole plane space about a ; it is, therefore, when the sides ac and ab are indefinitely produced, a finite part of that space. But the space between the parallels ab and de , extended indefinitely, can never, by any number of repetitions, be made to fill up the whole plane space, and it must therefore be an infinitely small part of that space. Hence, if ab and ac be sufficiently extended, the space between ab and ac must ultimately become greater than the space between ab and de , which requires that ac shall intersect de , and pass beyond it.

This method, we believe, was first suggested by one of the BERNOULLIS, and we see no objection to it, except that the idea of infinity

is involved in the reasoning ; and yet this same idea meets us in the very definition of parallel lines in the theory of the circle, and in incommensurable quantities.

The mention of incommensurables brings us to our last remark : it is, that LEGENDRE'S method of treating these quantities leaves the learner with very vague and indistinct ideas of them. We have two angles, A and B , and their arcs C and D , and we prove the relation $\frac{A}{B} = \frac{C}{D}$. If the learner asks what $\frac{A}{B}$ and $\frac{C}{D}$ mean, we tell him that all ratios are essentially and necessarily *numbers*, and cannot be anything else ; but if A and B are incommensurable, $\frac{A}{B}$ cannot be expressed by any conceivable number, nor can $\frac{C}{D}$; we may say that whatever expresses $\frac{A}{B}$ will also express $\frac{C}{D}$; but he may reply that no number can be made to express either, and how shall we get out of the difficulty ? Simply by showing from the outset that incommensurable quantities approximate to commensurable as their divisor is made smaller and smaller, and that as the relation $\frac{A}{B} = \frac{C}{D}$ subsists at each step in the approximation, it will continue to be true, however far the process be carried ; and yet as $\frac{A}{B}$ and $\frac{C}{D}$ are at each step ratios between commensurable quantities, they are always actual numbers.

To illustrate : suppose $A = 3$ and $B = 5 + R$, then $C = 3$ and $D = 5 + r$; neglecting R and r $\frac{A}{B} = \frac{3}{5} = \frac{C}{D}$. Now take a divisor of A and B smaller than R ; C and D will be divided into parts smaller than r ; and let

$$A = 7 \text{ and } B = 12 + R', \quad C = 7 \text{ and } D = 12 + r'.$$

Then neglecting R' and r' $\frac{A}{B} = \frac{7}{12} = \frac{C}{D}$. And so on. — PIKE POWERS, Staunton, Va.

V. *Problem*. — In any triangle, plane or spherical, if α, β, γ denote the perpendiculars dropped from the angles upon the opposite sides,

ϱ, r the radii of the inscribed and circumscribed circles, δ the distance between the centres; $\varrho', \varrho'', \varrho'''$ the radii of the escribed circles, $\delta', \delta'', \delta'''$ the distances between the centres of these circles and the centre of the circle whose radius is r , prove that in the plane

$$\begin{aligned} (1) \quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} &= \frac{1}{\varrho}, & (2) \quad -\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} &= \frac{1}{\varrho'}, \\ (3) \quad \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\gamma} &= \frac{1}{\varrho''}, & (4) \quad \frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\gamma} &= \frac{1}{\varrho'''} \end{aligned}$$

Squaring these four equations, and adding, we get

$$(5) \quad \frac{4}{\alpha^2} + \frac{4}{\beta^2} + \frac{4}{\gamma^2} = \frac{1}{\varrho^2} + \frac{1}{\varrho'^2} + \frac{1}{\varrho''^2} + \frac{1}{\varrho'''^2},$$

Also, as is easily seen,

$$(6) \quad \frac{8}{\alpha\beta} = \frac{1}{\varrho^2} + \frac{1}{\varrho'''^2} - \frac{1}{\varrho'^2} - \frac{1}{\varrho''^2}, \quad \frac{8}{\alpha\gamma} = \frac{1}{\varrho^2} + \frac{1}{\varrho''^2} - \frac{1}{\varrho'^2} - \frac{1}{\varrho'''^2},$$

$$\frac{8}{\beta\gamma} = \frac{1}{\varrho^2} + \frac{1}{\varrho'^2} - \frac{1}{\varrho''^2} - \frac{1}{\varrho'''^2}.$$

In the sphere,

$$\begin{aligned} (1) \quad \frac{1}{\sin \alpha} + \frac{1}{\sin \beta} + \frac{1}{\sin \gamma} &= \frac{1}{\sin \varrho} \frac{\cos \delta}{\cos r}, \\ (2) \quad -\frac{1}{\sin \alpha} + \frac{1}{\sin \beta} + \frac{1}{\sin \gamma} &= \frac{1}{\sin \varrho'} \frac{\cos \delta'}{\cos r}, \\ (3) \quad \frac{1}{\sin \alpha} - \frac{1}{\sin \beta} + \frac{1}{\sin \gamma} &= \frac{1}{\sin \varrho''} \frac{\cos \delta''}{\cos r}, \\ (4) \quad \frac{1}{\sin \alpha} + \frac{1}{\sin \beta} - \frac{1}{\sin \gamma} &= \frac{1}{\sin \varrho'''} \frac{\cos \delta'''}{\cos r}. \end{aligned}$$

Equations corresponding to (5), (6) in the plane can be found in the same way. Analogies (2), (3), (4) are new to me, but it is hardly probable that so simple relations have not before been observed. — DAVID TROWBRIDGE, Perry City, N. Y.

VI. *Magic Table.* — The following curious and ingenious table is taken from an old newspaper, where it appears under the head of “*Magic Table to tell a Lady's Age.*”

1st.		2d.		3d.		4th.		5th.		6th.	
1		2		4		8		16		32	
1	33	2	34	4	36	8	40	16	48	32	48
3	35	3	35	5	37	9	41	17	49	33	49
5	37	6	38	6	38	10	42	18	50	34	50
7	39	7	39	7	39	11	43	19	51	35	51
9	41	10	42	12	44	12	44	20	52	36	52
11	43	11	43	13	45	13	45	21	53	37	53
13	45	14	46	14	46	14	46	22	54	38	54
15	47	15	47	15	47	15	47	23	55	39	55
17	49	18	50	20	52	24	56	24	56	40	56
19	51	19	51	21	53	25	57	25	57	41	57
21	53	22	54	22	54	26	58	26	58	42	58
23	55	23	55	23	55	27	59	27	59	43	59
25	57	26	58	28	60	28	60	28	60	44	60
27	59	27	59	29	61	29	61	29	61	45	61
29	61	30	62	30	62	30	62	30	62	46	62
31	63	31	63	31	63	31	63	31	63	47	63

The peculiarity of the table is this : If all the columns in which a particular number is found be indicated, the sum of the numbers standing at the head of such columns will equal the particular number. For instance, a particular number is found in the 1st, 2d, 4th, 5th, and 6th columns; what is the number? The sum $1 + 2 + 8 + 16 + 32 = 59$ is the required number. — ARTEMAS MARTIN, Franklin, Venango County, Pa.

VII. In Robinson's Algebra is found the following problem : — “ A man borrowed \$60; what sum shall he pay daily to cancel the debt, principal and interest, in 60 days; interest at 10 per cent for 12 months of 30 days each ? ” $\$1\frac{5}{800}$ and $\$1\frac{51}{200}$ are the answers given in different editions. Is either of them *correct*, or is the true answer $\$1\frac{1}{215}$? — D. W. HORT, Professor of Mathematics, New Hampton Institute, Fairfax, Vt.

Mathematical Monthly Notices.

Annual Report of the Board of Regents of the Smithsonian Institution, showing the Operations, Expenditures, and Condition of the Institution for the Year 1861.

This Report opens, as usual, with that of PROFESSOR HENRY, — the Secretary of the Institution, — giving the history of its transactions for the year; from which we learn that the eleventh volume of the "Smithsonian Contributions to Knowledge" is nearly ready, containing the following memoirs: On North American Oölogy, by DR. THOMAS M. BREWER; On the Total Eclipse of the Sun, September 7, 1858, observed near Olmos, Peru, by LIEUT. GILLIS; Discussion of the Magnetic and Meteorological Observations made at the Girard College Observatory, Philadelphia, from 1840 to 1845, by A. D. BACHE, LL. D.; A Memoir, by DR. JOHN LE CONTE, on the Coleoptera, or Beetles, known to inhabit the Middle and Eastern portions of the great Central Regions of temperate North America; Result of Magnetical and Hypsometrical Observations in Mexico, — to which is appended Notes on the Volcano of Popocatepetl and its vicinity; On the American Storm of December 20, 1836, and the European Storm of December 25, of the same year, by PROFESSOR ELIAS LOOMIS. The General Appendix of the Report contains: Four Lectures on Agricultural Chemistry, by PROFESSOR S. W. JOHNSON; Lectures on the Shells of the Gulf of California, by P. P. CARPENTER; Latest Researches of MAEDLER relating to the General Movement of the Stars around a Central Body; Report on the Transactions of the Society of Physics and Natural History of Geneva, from July, 1858, to June, 1859; Present State of Ethnology in relation to the form of the Human Skull; Memoir of PYRAMUS DE CANDOLLE; On the Means which will be available for Correcting the Measure of the Sun's Distance in the next twenty-five years; Reports on the State of Knowledge of Radiant Heat, made to the British Association for the Advancement of Science, at the Meetings in 1832, 1840, and 1854; Description of the Magnetic Observatory at the Smithsonian Institution; On the use of the Galvanometer as a Measuring Instrument; On Observation of Earthquake Phenomena; Meteorological Instruments; On filling Barometer Tubes; Account of the Construction of a Standard Barometer, and Description of the Apparatus and Processes employed in the Verification of Barometers at the Kew Observatory.

The following problems are from "The Lady's and Gentleman's Diary" for the year 1861: —

Question I. Given the sum of the two sides, the radius of the escribed circle touching the base externally, and either of the angles at the base, to construct the triangle. W. H. LEVY, Shalbourne.

II. If the distance between the angular points and the points of contact of the escribed circles of a triangle be in harmonical progression, the escribed radii will be in arithmetical progression, and *vice versa*. — THOMAS WILKINSON, Burnley.

III. If, from the angular points of the triangle ABC , two lines be drawn, making the same constant angles with the adjacent side, two triangles $A'B'C$, $A''B''C'$ are formed similar to the original; and the circles $AA'A''$, $BB'B''$, $CC'C'$ pass through the same point. Also, if r_1, r_2, r_3 be the radii of these circles, and R that of ABC ; then $r_1, r_2, r_3 = R$; and if O_1, O_2, O_3, O be the centres of the circles, $16 \Delta ABC \times \Delta O_1 O_2 O_3 = a^2b^2 + b^2c^2 + c^2a^2$ — EDWARD RÜTTER, Sunderland.

IV. Given the perpendicular, the vertical angle and the sum of the three sides to construct the triangle. — W. H. LEVY, Shalbourne.

V. From a small plane on the floor of a room the horizontal breadth of a window in a , and the angles of elevation of its top and bottom are β and β' respectively. Show that the direct illumination of the plane by the light of the sky through the window, is to its illumination by the whole sky, supposed of uniform brightness, as $x \sin (\beta + \beta') \sin (\beta - \beta')$ to 2π . — THOMAS DOBSON, B. A., Liverpool.

VI. PC, PC' , at right angles to each other, are tangents to an ellipse, and $CQ, C'Q$ the corresponding normals. Show that the locus of the intersection of CC', PQ bisects the ellipse. — STEPHEN WATSON, Haydonbridge.

VII. Let ABC be any triangle; O, O_1, O_2, O_3 the centres of its inscribed and escribed circles respectively. Then the twelve circles described about the component triangles of the complete quadrilaterals $O_1O_2OO_3, O_2O_1OO_3, O_3O_2OO_1$, will intersect, four and four, in A, B, C and their centres will lie, two and two, in six points on the circumference of the circumscribing circle. — T. T. WILKINSON, Burnley.

VIII. A circular target, for rifle-practice, weighing 18lbs., and measuring 2 feet in diameter is suspended from a horizontal axis, about which it is capable of revolving freely, the centre of the target being at a distance of 6 feet from the axis. Find at what distance from the axis it must be struck by a rifle ball, weighing 8 ounces, and moving with a given velocity, in order that the ball and target may move off together with the greatest possible velocity. — STEPHEN FENWICK, of the Royal Military Academy, Woolwich.

IX. O_2, O_3 are the centres of the escribed circles, touching the sides AC, AB , of the triangle ABC , externally. Join O_2B, O_3C , cutting the sides in P_1, Q_1 ; O_2Q_1, O_3P_1 , cutting them in P_2, Q_2 , and so on; then the distance of any point in the line P_nQ_n , from BC , is equal to n times the sum of its distances from AC, AB . Also the lines $P_1Q_1, P_2Q_2, \dots, P_nQ_n$, all pass through the same fixed point. — STEPHEN WATSON, Haydonbridge.

X. If from the angles of a triangle Δ , straight lines be drawn to the opposite sides, dividing them in the proportion of m to n , where $m + n = 1$, and a triangle Δ_1 be formed having the dividing lines as sides, show that $\Delta_1 = \Delta (1 - mn)$. — PETRARCH.

XI. If a straight line be drawn parallel to a given straight line, and on the same side of it with a given circle; generally four circles may be described through any point in the parallel to touch the given straight line and to touch the circle, two externally and two internally. The ratio of a mean proportional betwixt the radii of external contact to a mean proportional betwixt the radii of internal contact, will be the same, wheresoever on the parallel the point be taken; and if the parallel be drawn through the centre of the circle, it will be a ratio of equality. Required a geometrical proof. — REV. WM. MASON, Normanton, Yorkshire.

XII. Compare the times of the small oscillations of a rod of any given length, not exceeding the earth's diameter; 1st, when it is suspended by one extremity, the other just touching the earth's surface; 2dly, when it is suspended by one end at the surface, and vibrates in a diametrical perforation through the earth. — C. H. BROOKS, Great George Street, Westminster.

XIII. A small horizontal orifice is made at the bottom of a vessel containing fluid. Determine the shape of the vessel, so that the form of the jet, at the instant the vessel is empty, shall be a straight line. — REV. S. TEBAY, Rivington.

XIV. Three points being taken at random in space as the corners of a plane triangle, determine the probability that it shall be acute. — THE EDITOR.

XV. If an ellipse be inscribed in an equilateral triangle so as to have one of its foci on the inscribed circle, then the other focus will lie on the hypocycloid of three branches, whose cusps are at the angular points of the triangle. — "CONIC," of St. John's College, Cambridge.

Editorial Items.

MR. G. W. HILL, the author of the Prize Essay printed in this number of the Monthly, is a graduate of Rutgers College, New Brunswick, N. J., and at present a member of the Lawrence Scientific School, Cambridge, Mass., pursuing his Mathematical studies under the direction of PROFESSOR PEIRCE.

SIR JOHN HERSCHEL'S PORTRAIT. — In our next number we shall present to our readers a Steel Portrait of SIR JOHN HERSCHEL, engraved by H. WRIGHT SMITH, Esq., of Boston, from a recent photograph, which is pronounced an excellent likeness. The number will also contain a complete catalogue of the titles of the distinguished author's works, in all *two hundred and twenty-four*.

The following gentlemen sent solutions of Prize Problems in the October number: JOHN A. WEINBRENER, Princeton College, N. J.; ASHER B. EVANS, Madison University, Hamilton, N. Y.; E. O. GIBSON, South New Berlin, N. Y.; J. O. ELLINWOOD, and D. MARTIN, Liberal Institute, Clinton, N. Y.; THOMAS M. SECHLER, E. K. LEONARD, EUGENE B. WILLARD, Marietta College, Ohio; G. A. OSBORNE, JR., Lawrence Scientific School; G. W. LEWIS, Middlebury College, Vt.; WILLIAM TIMPSON, White Plains, N. Y.; A. G. BARKER, Waterville College, Me.; H. TEIMAN, Baltimore, Md.; JOSEPH H. BUCKLEY, Yale Scientific School; G. B. HICKS, Cleveland, Ohio; JOHN F. CURTIS, Piedmont Institute, Liberty, Va.

Solutions of the Problems in the November number have been received from the following gentlemen: EDWIN A. DODD, Gouverneur, N. Y.; W. D. MERSHON, Princeton College, N. J.; DAVID TROWBRIDGE, Perry City, N. Y.; G. A. OSBORNE, JR., Lawrence Scientific School; G. B. HICKS, Cleveland, Ohio; ASHER B. EVANS, Hamilton College, N. Y.; A. G. BARKER, Waterville College, Me.; H. TEIMAN, Baltimore, Md.; WILLIAM WOOLSEY JOHNSON, Yale College, New Haven, Ct.

BOOKS RECEIVED. — A Higher Arithmetic: embracing the Science of Numbers, and the Art of their Application. By A. SCHUYLER, M. A., Principal of Seneca C. Academy, and Professor (elect) of Mathematics in Baldwin University. New York: Sheldon & Co., 115 Nassau street. 1860. — Plane Trigonometry, for the Use of Colleges and Schools. With Numerous Examples. By I. TODHUNTER, M. A., Fellow and Principal Mathematical Lecturer of St. John's College, Cambridge. Second edition, revised. Cambridge: MacMillan & Co. 1860. — The Claim of LEIBNITZ to the Invention of the Differential Calculus. By DR. H. SLOMAN. *Translated from the German, with Considerable Alterations and New Addenda by the Author. Cambridge: MacMillan & Co. — DODD'S Mathematical Text-books. Elementary and High School Arithmetics, 2 vols.; Elementary and High School Algebras, 2 vols.; Elements of Trigonometry, Plane and Spherical; with Applications to Heights and Distances, the Areas of Polygons, Surveying, Navigation, and the Solution of Astronomical Problems; Elements of Geometry and Mensuration. New York: Published by Pratt, Oakley, & Co.

Annual Report of the Board of Regents of the Smithsonian Institution, showing the Operations, Expenditures, and Condition of the Institution for the year 1859. Washington. 1860. — The Quarterly Journal of Pure and Applied Mathematics. No. 14. October, 1860. London: John W. Parker & Son. — Nouvelles Annales de Mathématiques. Tome Vingtième. Augmenté d'un Bulletin de Bibliographie, d'Histoire et de Biographie Mathématiques. Janvier, 1861. Paris: Mallet-Bachelier.

THE
MATHEMATICAL MONTHLY.

Vol. III . . . APRIL, 1861. . . No. VII

PRIZE PROBLEMS FOR STUDENTS.

I. Let ACB , ADB be two right-angled triangles having a common hypotenuse, AB ; join CD , and on CD produced both ways draw perpendiculars AE , BF . Show that

$$CE^2 + CF^2 = DE^2 + DF^2.$$

II. A clock gains $3\frac{1}{4}$ minutes in 15 seconds under the 24 hours. At noon it is two minutes slow; when will it indicate correct time?

III. Two railroads intersect at an angle φ ; and on these roads, distant from the junction a and b miles respectively, are two trains running towards the junction at the rates of m and n miles respectively. When and where will these trains be at a minimum distance from each other? — Communicated by Prof. WRAY BEATTIE, Wesleyan University, Mt. Pleasant, Iowa.

IV. The former of the two similar triangles $A'B'C'$ and ABC is circumscribed about the triangle DEF , and the latter is inscribed in DEF . Show that DEF is a mean proportional between $A'B'C'$ and ABC only when the homologous sides of the latter are parallel; also determine when ABC is a minimum, $A'B'C'$ and DEF being given. — Communicated by ASHER B. EVANS, Madison University, Hamilton, N. Y.

V. If r be the radius of the sphere inscribed in a tetrahedron formed by three rectangular co-ordinate planes and the plane

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, and d be the distance of this plane from the origin, prove that $\frac{1}{r} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$.

Solutions of these problems must be received by June 1st, 1861.

AWARD OF THE PRIZES FOR SOLUTIONS OF PROBLEMS
IN No. III., Vol. III.

The first prize is awarded to GEORGE B. HICKS, Cleveland, Ohio.

The second prize is awarded to ASHER B. EVANS, Madison University, Hamilton, N. Y.

The third prize is awarded to DAVID TROWBRIDGE, Perry City, N. Y.

The fourth prize is awarded to W. W. LEMMON, M'Kendree College, Lebanon, Illinois.

PRIZE SOLUTION OF PROBLEM I.

By W. W. LEMMON, M'Kendree College, Lebanon, Ill.

From AC , the diagonal of a square, $ABCD$, cut off AE equal to one fourth of AC , and join BE , DE . Show that the figure $BADE$ equals twice the square on AE .

The triangles ADE and ABE are equal, because they have the same base, AE , and the equal altitudes; and each is equal to one fourth of ADC or ABC . Therefore $4BADE = AB^2$; and since $AC^2 = 2AB^2$, we have

$$BADE = \frac{1}{4} AB^2 = \frac{1}{8} AC^2 = \frac{1}{8} (4AE)^2 = 2AE^2.$$

Mr. WILLIAM TIMPSON says: Since the semi-diagonal AF is bisected in E , EG parallel to FB bisects AB in G . The triangle

$$AEG = \frac{1}{2} AE^2 = GEB;$$

hence $AEB = AE^2$; $\therefore BADE = 2AE^2$. Or, complete the square $FECH$. Draw BE , cutting GH in L , and produce it to meet DA in M . The triangles EGL , BHL being equal, the triangle

$$BFE = BEA = EF^2 = AE^2,$$

as before.

Corollary. — The triangles AME , BCE being similar,

$$(EC = 3 AE) : AE = EB : EM = (BC = AD) : AM;$$

or $3 : 1 = EB : EM = AD : AM; \therefore AM = \frac{1}{3} AD;$

and $4 : 1 = (EB + EM = BM) : EM; \therefore EM = \frac{1}{4} BM.$

PRIZE SOLUTION OF PROBLEM II.

By E. S. STEARNS, Chester, Morris Co., N. J.

Two toothed wheels work against each other. Show that, if the number of teeth in one be prime to that in the other, before two teeth, which have once been in contact, come in contact again, every tooth of the one wheel will have been in contact with every tooth of the other.

Let $a > b$, prime to each other, be the number of teeth in the wheels A and B ; then, before any two teeth which are together can come together again, each wheel must make a whole number of revolutions; A makes b , and B makes a revolutions. Again, when B makes one revolution, every tooth of B has been in contact with one tooth of A , and in making a revolutions every tooth of B has been in contact with a , or every tooth of A .

PRIZE SOLUTION OF PROBLEM III.

By GEORGE B. HICKS, Cleveland, Ohio.

Eliminate φ from the equations

$$\begin{aligned} x \cos (\varphi + \alpha) + y \sin (\varphi + \alpha) &= a \sin 2 \varphi, \\ y \cos (\varphi + \alpha) - x \sin (\varphi + \alpha) &= 2 a \cos 2 \varphi, \end{aligned}$$

showing that

$$(x \sin \alpha - y \cos \alpha)^2 + (y \sin \alpha + x \cos \alpha)^2 = (2 a)^2.$$

Develop the left-hand members of these equations, and divide the first by the second; we get

$$\frac{x \cos \alpha + y \sin \alpha + \tan \varphi (y \cos \alpha - x \sin \alpha)}{y \cos \alpha - x \sin \alpha - \tan \varphi (x \cos \alpha + y \sin \alpha)} = \frac{\tan 2 \varphi}{2} = \frac{\tan \varphi}{1 - \tan^2 \varphi}.$$

$$\therefore \tan^3 \varphi = \frac{x \cos \alpha + y \sin \alpha}{y \cos \alpha - x \sin \alpha} = \frac{m}{n}.$$

$$\therefore \sin \varphi = \frac{m^{\frac{1}{2}}}{(m^{\frac{1}{2}} + n^{\frac{1}{2}})^{\frac{1}{2}}}, \quad \cos \varphi = \frac{n^{\frac{1}{2}}}{(m^{\frac{1}{2}} + n^{\frac{1}{2}})^{\frac{1}{2}}}.$$

But the first equation may be written

$$\frac{m}{\sin \varphi} + \frac{n}{\cos \varphi} = 2a = (m^{\frac{1}{2}} + n^{\frac{1}{2}})^{\frac{1}{2}}.$$

Hence, observing that $n^{\frac{1}{2}}$ and $(-n^{\frac{1}{2}})$ are identical, we obtain

$$m^{\frac{1}{2}} + n^{\frac{1}{2}} = (x \sin \alpha - y \cos \alpha)^{\frac{1}{2}} + (y \sin \alpha + x \cos \alpha)^{\frac{1}{2}} = (2a)^{\frac{1}{2}}.$$

This elegant solution is the only complete one received, owing doubtless to the misprint of a sign in the problem.

PRIZE SOLUTION OF PROBLEM IV.

By ASHER B. EVANS, Madison University, Hamilton, N. Y.

Show that $\frac{1}{4}m^2\sqrt{3}$ is the area of the greatest triangle which can be formed with the lines a, b, c , subject to the condition $a^3 + b^3 + c^3 = 3m^3$.

If A denotes the area of the triangle, then

$$u = 16A^2 = 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4$$

must be a maximum subject to the given condition. Therefore

$$(1) \quad \frac{1}{4}du = (ab^2 + ac^2 - a^3)da + (ba^2 + bc^2 - b^3)db \\ + (ca^2 + cb^2 - c^3)dc = 0,$$

$$(2) \quad a^2da + b^2db + c^2dc = 0.$$

Multiply (2) by λ , an indeterminate multiplier, add the product to (1), and equate the coefficients of the arbitrary differentials da, db, dc , to zero, which gives

$$a(b^2 + c^2 - a^2 + \lambda a) = 0, \quad b(a^2 + c^2 - b^2 + \lambda b) = 0, \\ c(a^2 + b^2 - c^2 + \lambda c) = 0.$$

In these three equations, put $b = na, c = ra$, and they become

$$\begin{aligned}(n^2 + r^2 - 1)a + \lambda &= 0, & (1 + r^2 - n^2)a + \lambda n &= 0, \\ (1 + n^2 - r^2)a + \lambda r &= 0.\end{aligned}$$

By adding the first and second, first and third, and second and third of these equations, we get

$$2r^2a + \lambda(1+n) = 0, \quad 2n^2a + \lambda(1+r) = 0, \quad 2a + \lambda(r+n) = 0;$$

$$\therefore \lambda = -\frac{2r^2a}{1+n} = -\frac{2n^2a}{1+r} = -\frac{2a}{r+n},$$

$$\therefore r^2 = n^2 + n^2 - r^2 = n + 1 - r^2n; \quad \therefore \{(n+1)^2 + r^2\}(n-1) = 0;$$

$$\therefore n-1 = 0, \quad \text{and} \quad (n+1)^2 + r^2 = 0.$$

Since the sum of the squares of two real quantities cannot be zero, the only real value of n is unity, which gives $r = 1$. Therefore $a = b = c$, and from the given condition

$$a = b = c = m; \quad \therefore A = \frac{1}{4}m^2\sqrt{3}.$$

Mr. TROWBRIDGE shows that $A = \frac{1}{4}m^2\sqrt{3}$ is also the maximum triangle when $a^n + b^n + c^n = 3m^n$.

PRIZE SOLUTION OF PROBLEM V.

The equation of a family of ellipses is $ax^2 + by^2 = 1$, when $a - b = c$, a constant. Show that the curve, which cuts all the individuals at an angle whose tangent is $\frac{y}{x}$, is $x = Ce^{\frac{cy^2}{2}}$.

Let $\theta, \theta_1, \theta_2$ be the angles made by the tangents to the ellipse, and trajectory with each other and with the axis of x . By trigonometry, we have

$$(1) \quad \tan \theta = \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2},$$

since $\tan \theta = \tan(\theta_1 - \theta_2)$.

But $\tan \theta_1 = \frac{dy}{dx} = -\frac{ax}{by}$ form the equation to the family of

ellipses, and $\tan \theta_2 = \frac{dy}{dx}$ as referred to the trajectory; and since $\tan \theta = \frac{y}{x}$, (1) becomes, by substitution and reduction,

$$(ax^2 + by^2)dx - (a-b)xydy = 0;$$

$\therefore \frac{dx}{x} - cydy = 0$, or, by integration, $\log x - \frac{cy^2}{2} = \log C$, or, passing to exponentials, $x = Ce^{\frac{cy^2}{2}}$.

This solution was substantially given by all the competitors.

SOLUTION OF NUMERICAL EQUATIONS OF THE THIRD DEGREE.

By PROF. J. C. PORTER, Liberal Institute, Clinton, N. Y.

THE following method of obtaining the approximate roots of cubic equations is analogous to HORNER'S, in being based upon the principle of successive transformations. But in obtaining the derived polynomials the common method of synthetical division is departed from, the trial divisors, corrections, and complete divisors being expressed by formulas, which give to the operation a modified form. This modification is supposed to possess the following advantages:—

1. The operation is somewhat abbreviated, even in the absence of decimal contraction.
2. The application of decimal contraction is more perfect, especially in equations having extended decimal coefficients.
3. In the several columns, the terms which correspond to any figure of the root are all situated between the same two horizontal lines, — an arrangement which facilitates a review of the work and the correction of mistakes.

Let us assume, for the general form,

$$(1) \quad y^3 + sy^2 + py = q;$$

and put $y = m + r$, in which m represents the initial figure or figures of the root, and r the remaining part. Substituting this value of y in equation (1), and factoring with reference to r , we have,

$$(2) \quad (m^3 + sm^2 + pm) + (3m^2 + 2sm + p)r + (3m + s + r)r^2 = q.$$

$$\begin{aligned} \text{Let} \quad \delta &= m^3 + sm^2 + pm, \\ \delta' &= 3m^2 + 2sm + p, \\ \delta'' &= 3m + s + r; \end{aligned}$$

then equation (2) becomes

$$\delta + \delta' r + \delta'' r^2 = q;$$

or,

$$(3) \quad \delta + (\delta' + \delta'' r)r = q.$$

From this equation we perceive that, if δ be subtracted from the absolute term, q , the remainder may be divided by δ' , as a trial divisor, to obtain r , the next figure of the root; the divisor may then be completed by the addition of $\delta'' r$. It is evident, also, that the new trial divisor for obtaining r_1 , the next figure of the root, will be what δ' will become when $m + r$ is substituted for m . This we find to be

$$\begin{aligned} (3m^2 + 2sm + p) + 2(3m + s + r)r + r^2 &= \delta' + 2\delta'' r + r^2 \\ &= \delta' + \delta'' r + (\delta'' r + r^2). \end{aligned}$$

To correct this trial divisor we must use, instead of δ'' , what δ'' becomes when $m + r$ is substituted for m , and r_1 for r ; this we find to be

$$3m + 3r + s + r_1 = 3m + r + s + (2r + r_1) = \delta'' + (2r + r_1).$$

To recapitulate, we have,

$$\begin{aligned} \delta &= \text{first subtrahend.} \\ \delta' &= \text{trial divisor.} \end{aligned}$$

δ''	= first factor of correction.
$\delta'' r$	= correction.
$\delta' + \delta'' r$	= complete divisor.
$\delta' + \delta'' r + \delta'' r + r^2$	= new trial divisor.
$\delta'' + 2r + r_1$	= first factor of new correction.
$(\delta'' + 2r + r_1) r_1$	= new correction.

It will be seen by inspecting these formulas, —

1st. That the new trial divisor is formed by adding to the last complete divisor the last correction and the square of the last figure of the root.

2d. The first factor of the new correction is formed by adding to the first factor of the former correction twice the former figure of the root plus the new figure. In all cases *algebraical* addition is understood.

Let it be required to find one value of y in the equation

$$y^3 - 4y^2 + 5y = 12.$$

We find, by trial, the initial figure, $3 = m$; whence we obtain $\delta = 6$, and $q - \delta = 6.000$, the first dividend. We next obtain the trial divisor, $\delta' = 8.00$, and the second figure of the root, $.5 = r$. We then find $\delta'' = 5.5$, $\delta'' r = 2.75$, whence we obtain the complete divisor 10.75 , the subtrahend 5.375 , and a new dividend, $.625000$. Next, adding 2.75 , the last correction $.25$, the square of the last figure of the root, and 10.75 , the last complete divisor, we have 13.7500 , the new trial divisor. Having next obtained 4 , the third figure of the root, we annex it to twice the former figure, making 1.04 , which we add to the 5.5 , and thus obtain 6.54 , the first factor of the new correction; and so on, as in the following operation: —

		3.544 + , root.	
δ''	$\delta'' r$	δ'	
5.5	2.75	8.00	6.000
1.04	.25	10.75	5.375
6.54	.2616	13.7500	.625000
84	16	14.0116	.560464
6.624	.026496	14.274800	64536000
		14.301296	57205184
			7330816

In practice, it will be found that the location of the corresponding terms in columns $\delta'' r$ and δ' , admits of the requisite facility in addition.

To continue the operation by decimal contraction, we observe that in any application of decimal contraction every multiplicand should have at least one redundant figure; that is, one place more than is necessary to obtain the required number of places in the product. This figure may be multiplied mentally, the tens of the product being carried to the product of the next figure, according to the common rule for contracted decimal multiplication. The suggestions which follow will enable us to secure this condition.

1st. For each new figure in the root, the terms in the operation extend to the right 1 place in the column marked δ'' , 2 places in the column marked δ' , and 3 places in the column of dividends. Hence,

2d. If at any point we omit to extend the dividend to a new period, we must *shorten* the corresponding term in column δ' 1 place, and the corresponding term in column δ'' 2 places.

3d. If, however, for the first contraction in columns δ' and δ'' , we simply omit the *extended parts*, and afterward contract according to the precept just given, each contracted multiplicand will have *one redundant figure*.

It will be observed, that, after the contraction commences, the square of the last figure in the root is not used in forming the new trial divisors, for it is of an order inferior to the lowest in the divisor to be found.

In accordance with these suggestions, the entire operation will appear as follows, decimal points being omitted.

		3.5445115283 $\pm$, root.	
		12	
δ''	$\delta'' r$	δ'	6
55	275	800	6000
104	25	1075	5375
654	2216	137500	625000
84	16	140116	560464
6624	26496	14274800	64536000
85	16	14301296	57205184
6632	3316	14327808	7330816
		14331124	7165562
66	7	1433444	165254
		1433451	143345
		143346	21909
			14335
		14335	7574
			7168
		1434	406
			287
		143	119
			114
		14	5
			4

Let it now be required to find one root of the equation,

$$y^3 + .34785634 y^2 - 1.03562855 y = -65.85732648.$$

In examples like this it will be convenient to write the terms in column $\delta'' r$ with units of like order in the same column.

Having found $m = -4$, and computed $\delta = -54.29178436$, and $\delta' = 44.18152073$, the operation will appear as follows :—

		—4.245574430 $\pm$, root.	
		—65.85732648	
		—54.29178436	
δ''	$\delta''r$	δ'	
—11.85214366	+2.37042873	+44.18152073	—11.56554212
.44	4	46.55194946	9.31038989
12.292144	.4916858	48.9223782	2.25515223
85	16	49.4140640	1.97656256
12.3771	.061886	49.907350	.27858967
105	25	49.969236	.24984618
12.39	.00620	50.03115	2874349
		50.03735	2501868
12.	.0008	50.0436	372481
		50.0444	350311
		50.045	22170
			20018
		50.05	2152
			2002
		50.1	150
			150
		50.	0

The root thus obtained will always be correct to as many places as are found in any term in the column of dividends.

GEOMETRICAL METHOD.

The formulas for these operations may be deduced geometrically, by a process very similar to that required for the corresponding rule for cube root. For this purpose, let the following problem be proposed :—

Required the dimensions of a rectangular parallelopipedon whose breadth and length exceed its height by d and d' respectively, and whose solidity is g .

Let y = the height; then $y + d$ = the breadth, and $y + d'$ = the length. Multiplying together the three dimensions, we have

$$(1) \quad y^3 + (d + d')y^2 + dd'y = q.$$

Put $s = d + d'$, and $p = dd'$; then

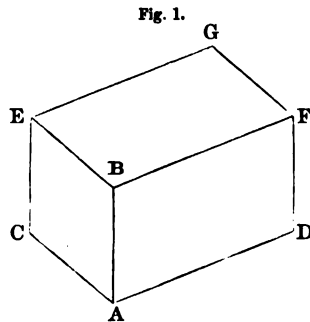
$$(2) \quad y^3 + sy^2 + py = q,$$

the general form employed in the foregoing pages.

We will suppose the required root of this equation to be irrational, and let m = the initial figure. Since the entire root is to be the height of a right prism whose several dimensions are as given in the problem, we will form a right prism (Fig. 1), having $AB = m$, the height; $AC = m + d$, the breadth; and $AD = m + d'$, the length. The solidity of the prism is

$$m(m + d)(m + d') = m^3 + (d + d')m^2 + dd'm = m^3 + sm^2 + pm = \delta.$$

Subtracting the solid contents of the prism from that of the required prism, we have, for a remainder, $q - \delta$. If we now enlarge our



prism (Fig. 1) by the addition of this remainder, its solidity will be that of the required prism. To preserve the given relations, the addition must be made upon three adjacent sides or faces, in the form of a covering of uniform thickness. The addition will therefore be composed of three flat prisms to cover the three faces (Fig. 2), and three oblong prisms and one cube to

fill the vacancies at the edges and corner (Fig. 3). Now the thickness of this enlargement will be additional height of the prism, and consequently the remaining part of the root. Hence, to obtain a second figure of the root, we may divide the remainder, $q - \delta$, by the sum of the areas of the three flat prisms (Fig. 2), as a *trial divi-*

sor. These areas are the same as the three faces of Fig. 1, and are as follows :—

$$\begin{aligned} ABEC &= m(m+d) &&= m^2 + dm. \\ ABFD &= m(m+d') &&= m^2 + d'm, \\ BFG E &= (m+d)(m+d') = m^2 + (d+d')m + dd'. \end{aligned}$$

Adding these areas, we find the sum to be

$$3m^2 + 2sm + p = \delta'.$$

Represent by r the figure obtained by this trial divisor, which will express the common width of the four remaining prisms (Fig. 3).

Fig. 2.

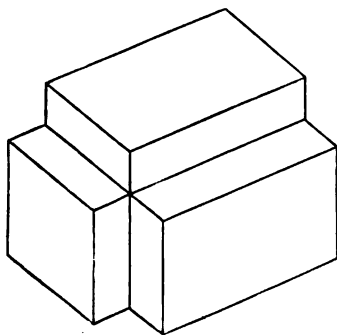
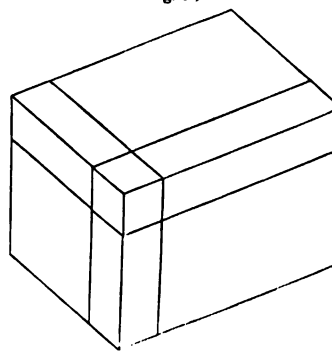


Fig. 3.



To complete the divisor, we must add to the trial divisor the sum of the areas of these four remaining prisms. The aggregate length of these prisms is $m + (m+d) + (m+d') + r = 3m + s + r = \delta''$; their area will therefore be $\delta''r$, which is the correction required.

Having completed the divisor, and formed a new remainder, we must proceed to make a new enlargement of the prism. The trial divisor to obtain the thickness of this second enlargement, or the next figure of the root, will be the area of three new flat prisms to cover the three faces of the prism already formed; and this surface (Fig. 3), is composed of one face of each of the former flat prisms, two faces of each of the oblong prisms, and three faces of

the cube at the corner. But we have in the last complete divisor one face of each of the flat prisms, oblong prisms, and cube; and in the last correction, one face of each of the oblong prisms and cube; and in the square of the last figure of the root, a third face of the cube; and these are the three quantities which were added together to form the new trial divisor, as before explained.

We have thus derived the algebraical formulas before used, by a process entirely geometrical. To adapt this explanation of the rule to the case of a negative root, we have only to give the prism the proper location relative to three rectangular co-ordinates, according to the principles of analytical geometry.

MOTION ON AN INCLINED PLANE.

(Prepared for Elementary Instruction.)

By THOMAS SHERWIN, Principal, English High School, Boston.

WE have seen, in the Theory of the Inclined Plane, No. V. Vol. II., that, when the power acts parallel to the plane, θ being the angle of the elevation, h , the height, and l , the length of the plane, R being 1, $P : W = \sin \theta : 1 = h : l$.

Now P , in this case, is equal to that part of the weight which urges the body down the plane. If, then, the body were unrestrained, it would, the elevation of the plane remaining the same, be subject to a constant force which would produce a uniformly accelerated velocity; and it is evident that this accelerating force bears to that of a body falling freely the same ratio that the equilibrating power bears to the weight. Therefore, calling g' the accelerating force down the plane, we have $g' : g = \sin \theta : 1 = h : l$; $\therefore$

$$(1) \qquad g' = g \sin \theta = \frac{h}{l} g.$$

Since the force which urges a body down a plane is the same as that which causes a body to fall freely, except that it is of less intensity, it follows that the motion of a body down such a plane is subject to the same laws as that of falling bodies. Hence, on the same plane,

$$(2) \quad t : t' = v : v',$$

$$(3) \quad s : s' = t^2 : t'^2 = v^2 : v'^2,$$

$$(4) \quad s_1 : s_2 : s_3, \text{ \&c.} = 1 : 3 : 5, \text{ \&c.};$$

that is,

The times are as the final velocities; the spaces are as the squares of the times, or as the squares of the final velocities; and the spaces described in equal successive portions of time are as the successive odd numbers, 1, 3, 5, 7, &c.

Since $g' = g \sin \theta = \frac{hg}{l}$, if we substitute these values of g' instead of g in the formulæ for falling bodies, we shall have corresponding formulæ for the motion of bodies on inclined planes. Hence,

$$(5) \quad v = gt \sin \theta = \frac{hgt}{l}; \quad (6) \quad t = \frac{v}{g \sin \theta} = \frac{lv}{hg}$$

$$(7) \quad s = \frac{1}{2}gt^2 \sin \theta = \frac{1}{2}\frac{hgt^2}{l}; \quad (8) \quad t = \sqrt{\frac{2s}{g \sin \theta}} = \sqrt{\frac{2ls}{hg}};$$

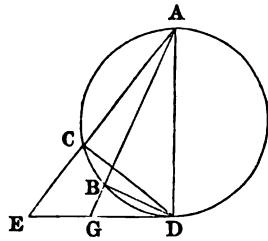
$$(9) \quad s = \frac{v^2}{2g \sin \theta} = \frac{lv^2}{2hg}; \quad (10) \quad v = \sqrt{2gs \sin \theta} = \sqrt{\frac{2hgs}{l}}.$$

Formulæ may be easily obtained for bodies *projected* down or up inclined planes. By reference to Articles IV. and V. of Falling Bodies, the student will find no difficulty in obtaining these formulæ.

II. Let u be the velocity which a body acquires in falling freely through the height h , and v the velocity which the body acquires in moving down l , the length of a plane. Then $u = \sqrt{2gh}$, and, from (10), since s equals l , $v = \sqrt{\frac{2hgl}{l}} = \sqrt{2gh}$; $\therefore u = v$. Hence,

A body acquires the same velocity in moving down the length of an inclined plane, as it would acquire in falling freely through the height of the plane.

Let t and t' be the times in which a body would descend the inclined planes AG and AE as far as to the perpendiculars BD and CD drawn from D to these planes, and let the elevation of these planes be θ and θ' . Then, since AB and AC are the spaces, we have



$$AB = \frac{1}{2} g t^2 \sin \theta, \quad \text{and} \quad AC = \frac{1}{2} g t'^2 \sin \theta'.$$

But, in the right-angled triangles ABD and ACD ,

$$AB = AD \cos BAD = AD \sin \theta,$$

and

$$AC = AD \cos CAD = AD \sin \theta';$$

$$\therefore AD \sin \theta = \frac{1}{2} g t^2 \sin \theta, \quad \therefore (1) \quad AD = \frac{1}{2} g t^2;$$

$$\text{and} \quad AD \sin \theta' = \frac{1}{2} g t'^2 \sin \theta', \quad \therefore (2) \quad AD = \frac{1}{2} g t'^2;$$

hence, from (1) and (2),

$$\frac{1}{2} g t^2 = \frac{1}{2} g t'^2, \quad \therefore t = t',$$

Hence, the times of descent through AB and AC are the same.

If upon AD as a diameter the circumference of a circle be described, it will pass through the vertices of the right angles B and C , and AB and AC are chords drawn from A . Hence :

If any number of chords are drawn from either extremity of the vertical diameter of a circle, a heavy body would move down these chords or fall freely through the vertical diameter in the same time.

III. If we call l and l' the lengths of two inclined planes, h and h' their heights, and t and t' the times of descent through these lengths since l and l' are the spaces, we have from formula (8), and from the analogous formula $t' = \sqrt{\frac{2 l' s}{h' g}}$, by putting l and l' for s ,

$$t = \frac{l \sqrt{2}}{\sqrt{h g}}, \quad \text{and} \quad t' = \frac{l' \sqrt{2}}{\sqrt{h' g}}; \quad \therefore t : t' = \frac{l \sqrt{2}}{\sqrt{g h}} : \frac{l' \sqrt{2}}{\sqrt{g h'}}.$$

Dividing both terms of the last ratio by $\sqrt{2}$, and multiplying by $\sqrt{g}$, we have

$$(1) \quad t : t' = \frac{l}{\sqrt{h}} : \frac{l'}{\sqrt{h'}}.$$

Hence,

In two different inclined planes, the times of descent through the lengths of the planes are as the lengths divided by the square roots of the heights.

If $h = h'$, proportion (1) becomes

$$(2) \quad t : t' = l : l'.$$

If in the same proportion $l = l'$, it becomes $t : t' = \frac{1}{\sqrt{h}} : \frac{1}{\sqrt{h'}}$;

or, multiplying both terms of the last ratio by $\sqrt{h}$ and $\sqrt{h'}$,

$$(3) \quad t : t' = \sqrt{h'} : \sqrt{h}.$$

Hence, from (2) and (3),

When in two inclined planes the heights are the same, the times of descent are as the lengths; and when the lengths are the same, the times of descent are inversely as the square root of the heights.

It may be proved in a similar way that the times of descent down the chords CD and BD are equal, and as the diameter itself may be regarded as a chord, the times of descent would each be equal to that in which a body would fall freely through this diameter.



THE THEORY OF THE GYROSCOPE.

By THEODORE G. ELLIS, Civil Engineer, Boston.

THE interest excited in this little instrument, by its wonderful power of seeming to overcome the force of gravity, has induced the following attempt at an explanation of its principles in as concise and simple a manner as possible, with a hope that it may call

out the investigations of others who have made experiments or researches upon this interesting subject.

To have an apparatus in every way suited to our investigations, with a view to their being more readily understood, let us suppose the ordinary Gyroscope, consisting of a wheel, axle, and supporting ring, to be hung in another ring by pivots at right angles to the direction of the axis of the wheel; and this second ring to be supported also by pivots, at right angles to the former ones, upon a stand capable of turning upon a vertical axis, which, if produced, would pass through the centre of the wheel. The lines joining the opposite points in the rings would also pass through the centre of the wheel.

We then have the Gyroscope hung in gimbals precisely like a mariner's compass, except the additional contrivance for turning upon a vertical axis.

This mode of hanging the wheel admits of its motion freely in all possible directions around its centre, except when the horizontal axis of the wheel, the vertical axis, and all the points of suspension lie in the same vertical plane. In this position, motion of the wheel round its centre in that plane cannot take place. In our present examination we will suppose this position not to occur, as in all others motion will be free in all directions around the centre of the wheel.

For the sake of simplicity, we will also suppose the revolving wheel to be a circular disk, suspended on an axis of no appreciable weight compared with the weight of the wheel.

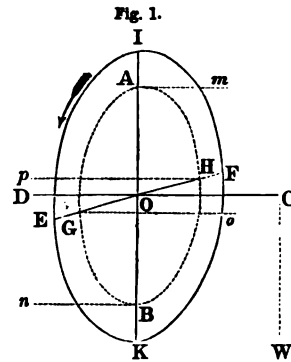
Let $IEKF$ (Fig. 1) be a circular disk, suspended as above described, and revolving with a given velocity in the direction of the arrow. Let DC and EF be horizontal, and IK vertical. At the end, C , of the axis, DC , let there be a force, W , acting downwards parallel to IK .

Then will each particle of matter in the wheel above the horizontal diameter, EF , have a tendency to move in the direction Am parallel to DC , and each particle of matter below EF will have a tendency to move in the opposite direction, Bn .

We see, therefore, that at any one instant of time, all the particles of matter above CF will be acted upon by a force in the direction Am , but when the wheel has performed a half revolution, these same particles are acted upon by a force in the direction Bn , exactly opposing and neutralizing the former. And so with the half circle below EF , it is at one instant acted upon in the direction Bn , and the next, when it has made half a revolution, it is acted upon in the direction Am . Thus each particle of which the wheel is composed receives in rapid succession impulses in opposite directions, which, owing to its inertia and the rigidity of the wheel, it constantly resists.

Let us take a particle in the vertical diameter, as at A , and follow it round the circle. During its passage through the quadrant AG it is continually acted upon by a succession of forces tending to move it in the direction Am , parallel to DC . After passing the horizontal diameter EF , from G to B it is acted upon by a succession of forces exactly equal to those above the horizontal diameter, but acting in an opposite direction, Bn , so that any motion or momentum acquired in passing through AG will be exactly counterbalanced and wholly destroyed in passing through GB , so far as relates to the directions Am and Bn ; so that if the particle was at rest at A , relative to those directions, it will be again at rest on arriving at B .

The same effects as above described also evidently take place while the particle passes through BHA .



What is true for any one particle must be true for every other, and consequently for the whole wheel, and the forces above and below EF just balancing each other, no motion around EF results from the force, W , while the disk is revolving with a uniform velocity.

Having thus shown that the force, W , will not produce motion about the horizontal diameter, EF , which is at right angles to DC , the axis, DC , will remain in a horizontal plane. Motion, therefore, of the plane, $IEKF$, can only occur about the vertical axis IK , as we have the point O fixed by construction, being the centre of all motion of which the wheel is capable.

Let us now examine the effect of the forces acting on any particle of matter passing around the circle, with regard to the axis IK .

From A to G we have seen that it is acted upon by a force tending to move it in the direction Am ; and from G to B by a force tending to move it in the direction Bn ; and that these forces are equal and act in opposite directions. Now if a body is acted upon by any force for a definite period of time, motion will be produced in the direction of the force, and a certain velocity will be acquired; if an equal force then act on the same body in an opposite direction, for the same period of time, the velocity acquired is gradually retarded and the body left at rest.

This is precisely the case with a particle of matter passing through AGB or through BHA . A force tends to accelerate its motion in the direction Am from A to G , and from G to B it is gradually retarded by a force acting in the direction Bn . So also through BHA a force tends to accelerate its motion in the direction Bn , while passing through BH , and through HA it is again retarded and brought to rest by the force acting in the direction Am .

As the forces producing these motions around IK are equal on opposite sides of IK , and their centres of moment are equally

distant, they will produce motion round IK as an axis independently of the point, O , being fixed by construction.

Let us for a moment consider what would be the effect if this circular motion around IK could not take place. Suppose, for instance, that the axis CD were confined to a vertical plane, and to be in a horizontal position. In this case no horizontal circular motion could occur, so that all particles of matter of which the wheel is composed passing through EO downwards and through OF upwards would be brought to relative rest with regard to a horizontal plane. But immediately on passing EO and OF they would be acted upon by the opposite forces in the directions Am and Bn ; motion would immediately take place around EF , and the weight W would fall.

We thus see that this motion around IK not only takes place, but is absolutely necessary to the equilibrium maintained at A and B .

We will now proceed to ascertain the time of rotation around IK . Suppose the circular disk to be divided into an indefinite number of small sectors, then will the centre of gravity of each be distant from the centre of the disk by two thirds of the radius of the disk. For convenience of demonstration we will consider all the weight of the disk to be in the circle joining these centres of gravity, calling the radius of this circle "radius of gravity," and limiting ourselves to such reasoning as will be consistent with the above supposition.

This method of considering the weight of the wheel to be in its circle of gravity has also the advantage of being applicable to wheels of all forms.

As the forces acting in the quadrants GB and HA are solely employed in bringing to a state of rest the motion acquired by each particle of matter in the quadrants AG and BH , they exercise no

influence upon the rotation of the wheel around IK , which rotation is therefore solely due to the motion acquired in the quadrants AG and BH .

Again, each particle of matter of which the wheel is composed is limited to the plane of revolution of the wheel, so that in passing through AG or BH it will not admit of being uniformly accelerated, but will move in successive instants of time through spaces proportional to the sine of the angular distance from IK . But as in passing through AG or BH each quantity of motion lost by one particle on account of its being confined to the plane of the wheel, is distributed among all others lying on the same side of IK , the total amount of motion for any one particle is the same as though it were uniformly accelerated while passing through AG or BH .

Referring again to Fig. 1, let

- r = AO = radius of gravity.
- a = OC .
- t = time employed in revolving.
- g = force of gravity = 32.2.
- π = ratio of circumference to diameter.
- W = force, in units of weight, at C .
- W' = weight of wheel.

To find the accelerating force, F , acting on each particle of the wheel and turning it round IK , we proceed as follows :—

We have W = number of units of weight acting at a radius of a tending to turn the wheel on GH . We have also the distance from O to the centres of moment of the semicircles

$$GAH \text{ and } GBH = \frac{2r}{\pi}.$$

Denoting by F the force in units of weight acting on the centres of moment of the semicircles GAH , GBH , we have

$$F \times \frac{2r}{\pi} = aW, \quad F = \frac{\pi a W}{2r}.$$

If then the weights W and W' were equal, we should have the accelerating force, F , acting on the wheel and the force of gravity, g , proportional to the pressures F and W , or

$$F : g = \frac{\pi a W}{2r} : W, \quad F = \frac{\pi g a}{2r}.$$

But if the weights W and W' are different, the force F will vary inversely as the proportion of W' to W , or directly as $\frac{W}{W'}$; hence,

$$F = \frac{\pi g a}{2r} \times \frac{W}{W'}.$$

By the law of bodies moving under the influence of a continuous force, the space described equals $\frac{1}{2} t^2 F$. We have, therefore,

$$(1) \quad s = \frac{1}{2} t^2 \times \frac{\pi g a W}{2r W'} = \frac{\pi g t^2 a W}{4r W'}.$$

Representing by T the time required for one whole rotation about IK , we have,

$$(2) \quad T : t = 2\pi r : \frac{\pi g t^2 a W}{4r W'},$$

$$T = \frac{8 r^2 W'}{g t a W}.$$

If n = number of revolutions of wheel per second, then $4t$, equal to the time of one revolution, $= \frac{1}{n}$, and $t = \frac{1}{4n}$. Substituting in above formula, we have

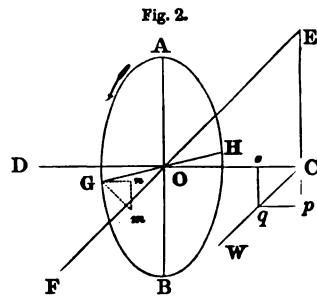
$$(3) \quad T = \frac{32 n r^2 W'}{g a W}.$$

In these formulæ the units of length and time are feet and seconds. Of weight, any unit may be taken.

Let us now consider the case where the axis CD is not horizontal, but inclined at any angle to the horizon except vertical, when the force W would pass through the centre O .

We will suppose the Gyroscope suspended as before, and revolving with a uniform velocity.

Let AB , CD , and GH (Fig. 2) represent the same lines as before, AB and CD lying in the same vertical plane, and GH being horizontal.



Let W be the weight hanging vertically from the end of the inclined axis CD . Draw the vertical line EF through the centre of the wheel, O . Represent the weight W by Cq , and resolve this force into two others lying in the vertical plane which passes through EF and CW ; one force being at right angles to, and the other coinciding with the axis CD . Completing the parallelogram we have one force $= Cp$, and the other $= Co$. Produce Cp to E .

The force Co acting through the centre of the wheel has no tendency to turn it in any direction. The force Cp , acting at right angles to CD , and tending to turn the wheel on the horizontal line GH , at right angles to CD , would, as we have before shown, cause the wheel to rotate on AB if it continued to act in a direction parallel to Cp . That is, the point G would move in the direction Gn . But this force is constantly changing its direction, as it always lies in a vertical plane passing through EF ; and as it constantly tends to turn the wheel in the vertical plane in which it lies, the line GH , being always at right angles to this plane, must remain horizontal, and the points G and H will move in a horizontal circle around O . The angle EOC will remain constant, and the wheel will rotate about the vertical line EF parallel to CW .

Let us now suppose the point C to revolve around EF the indefinitely small distance Cx . In passing through this indefinitely small space we may consider two forces to have been acting at the point

C , one of which is continually parallel to Cp , causing the wheel to rotate on AB , and the other operating to turn AB and GH on CD as an axis through the small angle AEx .

From G lay off Gn parallel to DC equal to the distance the point G would move under the influence of the parallel force Cp . Draw nm perpendicular to Gn , and make Gm horizontal. Then will Gn and nm represent the distances the point G would be moved by the two forces acting at the point C . Gm is the resultant, and represents the distance actually passed through by the point G .

In the triangles Gnm and Cqp we have the sides of one respectively perpendicular to the sides of the other in direction; therefore they are similar, and we have

$$Gn : Gm = Cp : Cq.$$

We see, from equation (1) previously obtained, that the space described in rotating by the influence of a force acting at right angles to CD is proportional to the weight W . If, therefore, Gn is the distance revolved under the influence of the force Cp , Gm would be the distance revolved under the influence of the force $Cq = W$ acting at right angles to CD , and is therefore the same distance that would be described by the point G when CD is horizontal.

As the proof of the above would be similar for any inclination below the horizontal line, we see that for all inclinations of CD we have the angular velocity and time of rotation about the vertical axis the same.

From the foregoing we may deduce the general law of the Gyroscope, which will account for all the phenomena exhibited by different experiments, viz. : —

A revolving body, acted upon by a force passing through its axis of revolution, will rotate with a new motion round an axis passing through its centre of motion parallel to the direction of the force.

The time of rotation will be directly proportional to the velocity of revolution, the square of the radius of gravity in the plane of revolution, and the weight of the body; and inversely proportional to the force, and the distance on the axis from the centre of the body to the line in which the force acts.

The case in which resistance occurs to the rotating motion comes under the above general law by considering the reaction from the resistance to be a force.

Let us see how this law operates in the ordinary form of toy gyroscope, apparently supporting its own weight and rotating around a fixed support.

Here we have the whole weight of the wheel and ring acting downwards by the force of gravity; but as the wheel does not yield to its influence, on account of its being obliged to turn out of its revolving plane to fall, which motion we have seen is resisted by a revolving body, the whole weight is transferred to the point of support, which reacts in an upward direction with an equal force to sustain it. Here we have a force acting on the end of the axle, equal to the weight of the wheel and ring, tending to turn the revolving wheel on its horizontal diameter. This we have seen will cause it to rotate on a vertical axis. But as the point of suspension is fixed, the rotation of the wheel on a vertical axis reacts on the point of support, and the wheel itself moves instead of the point at which the upward force acts. And as one rotation of the wheel on its vertical axis will cause one complete turn around the point of support, the time employed in making the latter will be given by the formula (3) previously found, viz. : —

$$T = \frac{32 \pi r^2 W'}{g a W}.$$

In this case W is the weight of the wheel and ring, and W' the weight of the wheel alone.

When the Gyroscope is suspended by a cord instead of the fixed support, it also appears to turn about the point of suspension. The reason of this is, that the whole weight of the apparatus being transferred as before to the point of suspension, that point is in stable equilibrium only when the cord hangs vertically, and the resistance to motion, to any distance out of this vertical line, is very much greater than the resistance of the air to the passage of the revolving wheel, so that the slight reaction on the point of suspension necessary to cause rotation round this point will move it but a small distance, and cause the cord to vary but slightly from a vertical position.

Finally, let us examine into the cause of the gradual falling of the weight attached to the axis of the Gyroscope.

We have already seen that when the wheel revolves uniformly and without friction, there is no yielding in the direction of the force, it being entirely expended in producing the rotation round the vertical axis.

That the falling of the weight is due to friction, including under this term all causes of resistance, may be shown as follows.

Let us consider the friction as operating in two ways: First, by causing a retardation of the velocity of the revolving wheel, and, secondly, by the resistance it offers to motion at the point of support.

It is evident, from our foregoing reasoning, that if a particle of matter is longer in passing through the second quadrant on either side of AB than through the first, it will more than have its motion arrested, and will acquire a motion in an opposite direction, and will not be at rest on arriving at A or B ; and by just so much distance as it has moved through will the wheel be turned on its horizontal diameter, thus allowing the weight to fall through a corresponding angle.

We have also seen that any force arresting the rotation about AB causes the weight to fall. The friction upon the point of support, though not actually stopping the rotation of the wheel, operates in the same manner. The particles of matter in the first quadrant are prevented from acquiring the full amount of motion due to the force, and thus in the second quadrant the motion is more than counterbalanced, so that instead of being at rest at the points A and B , motion is acquired in the opposite direction; and by just the distance moved through, after coming to rest before arriving at A or B , will the wheel be turned on its horizontal diameter and the weight will fall by a corresponding amount.

A COMPLETE CATALOGUE OF THE WRITINGS OF SIR JOHN HERSCHEL.

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- 1812. NICHOLSON'S Journal, XXXI. p. 133, No. 142. Analytical Formulæ by "A Lover of the Modern Analysis."
- 1812. Ibid., XXXII. p. 13. A letter signed "Analyticus." (Trigonometrical Formulæ.)
- 1812. LEYBOURNE'S Mathematical Repository for 1814, Part II p. 58. New Properties of the Conic Sections.
- 1812. Phil. Trans. Royal Soc. London, 1813. On a Remarkable Application of COTES'S Theorem.
- 1813. On Trigonometrical Series, ... &c., ... Mem. Analyt. Soc. Cambridge.
- 1813. On Equations of Differences and their Application to the Determination of Functions under Given Conditions. Mem. Analyt. Soc. Cambridge. N. B.—The latter portion of this Part III. on Functional Equations is all wrong.
- 1814. Phil. Trans. Royal Soc. London, 1814. Consideration of Various Points of Analysis.
- 1814. Problems 362, 363, 364, 365, 366, 367, 368, 369, LEYBOURNE'S Mathematical Repository. — Also (I believe somewhere in that Repository), Properties of the Catenary, analogous to the Trigonometrical Formulæ in the Circle, and a number of other problems, &c., in other parts of that Repository.
- 1815. Phil. Trans. Royal Soc. London, 1816. On the Development of Exponential Functions together with several new Theorems relating to Finite Differences.
- 1816. The notes N, O, P, Q, in the Translation of LACROIX'S 8vo Treatise on the Differential and Integral Calculus, by MESSRS. PEACOCK, HERSCHEL, and BABBAGE, with an original Appendix on Differences.
- 1817. The article "Isoperimetrical Problems" in the Edinburgh Cyclopædia.
- 1818. The article "Mathematics in the Edinburgh Cyclopædia, XIII. Parts 1, 2.
- 1818. On Circulating Functions, and on the Integration of a Class of Equations of Finite Differences into which they enter as Coefficients. Read before the Royal Soc., Feb. 19. Phil. Trans.
- 1819. The Preface to SPENCE'S Mathematical Essays. (Edited by Sir J. H.)
- 1819. Original Notes on SPENCE'S Mathematical Essays, published in 1819.
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- 1819. A Notice in BREWSTER and JAMIESON'S Journal of Sir W. HERSCHEL'S Paper on the Distance of the Fixed Stars.

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1819. On the Application of a new Mode of Analysis to the Theory and Summation of certain extensive Classes of Series. BREWSTER and JAMIESON's Edinburgh Phil. Journal, Vol. II. p. 23.
1819. Some Additional Facts relating to the Habitudes of the Hyposulphurous Acid and its Union with Metallic Oxides. BREWSTER and JAMIESON's Phil. Journal, Vol. II. p. 154.
1819. Refractive Indices of Hyposulphite of Soda and Silver and of Nitrate of Lead. BREWSTER and JAMIESON's Phil. Journal, Vol. II. p. 184.
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1826. On the Parallax of the Fixed Stars. Read before the Royal Society, March 9 and 16. Phil. Trans.

1827. Correction of an Error in a Paper on the Parallax of the Fixed Stars. Phil. Trans. Royal Soc.
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1830. Extract of a Letter from Sir J. F. W. H. to M. QUETELET, on the "Roccellic Acid." Read at the Royal Academy of Sciences, Brussels, May 22d, 1830. Bulletins of the Academy.
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1833. A Letter addressed to R. PHILLIPS, Esq. &c., &c., dated January 12th, 1833, on a Remarkable

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1833. On the Absorption of Light by Colored Media, viewed in connection with the Undulatory Theory. From the London and Edinburgh Phil. Magazine and Journal of Science.
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1834. On the Satellites of Uranus. Read March 14th, 1834. Royal Astron. Soc., Vol. VIII.
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1836. Report of the Meteorological Committee of the South African Literary and Philosophical Institution (published by the Institution).
1836. A Letter to Dr. ADAMSON of the South African College.
1836. Extract of a Letter to R. I. MURCHISON, Esq., on the *Primum Mobile* of Geological Disturbances. Printed by C. BABBAGE in his "Ninth Bridgewater Treatise," p. 214.
1836. On the Cape of Good Hope Government Schools. Printed in Mr. C. BAMBURY'S book on South Africa.
1836. Sixth Catalogue of Double Stars, observed at Slough, in the Years 1831 and 1832, with the Twenty-feet Reflector: containing the Places, Descriptions, and Measured Angles of Position of 286 of those Objects of which 105 have not been previously described. Reduced to the Epoch 1830. Memoirs Royal Astron. Soc., Vol. IX.
1837. A Letter to F. BAILY, Esq., dated Oct. 18th, 1836, on the Observations of the Comet of HALLEY, after the Perihelion Passage in 1836: made at Feldhausen, Cape of Good Hope. Read Jan. 13th, 1837. Trans. Royal Astron. Soc.
1837. Meteorological Observations at Feldhausen in March 21st, 1837 (Athenæum).
1837. " " " " " June 21st, 1837 "
1837. " " " " " Sept. 21st, 1837 "
1837. " " " " " Dec. 21st, 1837 "
1837. Report of the Meteorological Committee. Read July 17th, 1837.
1838. Astron. Nachr., No. 354. Extract of a letter to Dr. MÄDLER on the sudden Change of Magnitude of the star η Argus.
1838. Report of British Association, 1838. "Notices of Observations of Stars and Nebulae observed at the Cape of Good Hope."
1838. Notice on HALLEY'S Comet. Report of British Association, 1838.
1838. On a Method of Obtaining a great Increase of Light from a common Argand Burner. London, Edinburgh, and Dublin Phil. Magazine.
1839. Astron. Nachr., No. 372. Letter to Professor SCHUMACHER on the Variable Stars η Argus, α Hydræ, and α Cassiopeiæ; and on the comparative brightness of the Stars.
1839. Note on the Art of Photography; or on the Application of the Chemical Rays of Light to the purposes of Pictorial Representation. Notices of the Proceedings of the Royal Society of London. No. 37, p. 131.
1839. On the Variability of the Star α Cassiopeiæ. Notices of Royal Astron. Soc. May 10th, 1839.
1839. A Letter to the President of the British Association, giving an Account of the Production of analogous Colors by the Action of the colored prismatic Rays on Argentine Papers; and on a peculiar Property of the extreme red Rays. (Report of Proceedings of the British Association, 1839. Also Athenæum, No. 641.)

1839. Astron. Nachr., No. 405. "Requiem of the Old Telescope," with a Translation into German by Madame MADLER.
1840. Address to the Astron. Soc. on occasion of delivery of a Medal to M. PLANA. (Notices of Astron. Soc., Feb. 1840.)
1840. On the Chemical Action of the Rays of the Solar Spectrum on Preparations of Silver and other Substances, metallic and non-metallic. Phil. Trans. Royal Soc. London, 1840.
1840. Report on the Subject of Terrestrial Magnetism in a Series of Resolutions, adopted by the British Association, at their Meeting in August, 1838, at Newcastle. Report of the British Association for the Advancement of Science for 1839.
1840. On the Variability and Periodical Nature of the Star α Orionis. Read Jan. 10th, 1840. Trans. Royal Astron. Soc.
1840. Review of GAUSS and W. WEBER's allgemeine theorie des Erdmagnet; and of GAUSS's Intensitas vis Magnetice. Quarterly Review, No. 131. June, 1840.
1840. Report of the Committee of Physics, including Meteorology. Royal Soc. London.
1841. A Letter to Professor STEVELLY, Sec. British Association, "On Processes for producing Colored Photographs." British Association, 1841.
1841. On Sir WM. HAMILTON's Observations of the Meteors of Aug. 9th, during the great Eruption of Vesuvius described by him. Athenæum, No. 721. Aug. 21st, 1841.
1841. An Address delivered on the Presentation of the Gold Medal of the Royal Astronomical Society to Professor BESSEL, at the Anniversary Meeting, Feb. 12th, 1841, for his Observations and Researches on the Parallax of 61 Cygni.
1841. On the Advantages to be attained by a Revision and Re-arrangement of the Constellations, with especial Reference to those of the Southern Hemisphere; and on the Principles upon which such Re-arrangement ought to be conducted. Memoirs of the Royal Astron. Soc. Vol. XII.
1841. Report of the Magnetic Committee of the British Association.
1841. Astron. Nachr., No. 442, On the Moon's Influence on the Weather.
1841. Remarks on certain Suggestions offered to the Commission for the Restoration of the lost Standard of Weights and Measures. Report of the Commission, 1841.
1841. Remarks on the Nomenclature of Coins, Weights, and Measures, and on some other Points connected with the Subject of the Restoration of the lost Standards. Report of the Commission.
1841. A Review of WHEWELL's History and Philosophy of the Inductive Sciences. Quarterly Review, No. CXXXV.
1842. On the Action of the Rays of the Solar Spectrum on Vegetable Colors; and on some new Photographic Processes. Read June 16th, 1842. Phil. Trans., Part II., for 1842.
1842. The Walk. Translated in the original Metre from the German of F. SCHILLER. For private circulation.
1842. Extracts from a letter addressed by Sir J. F. W. HERSCHEL to Mr. MURCHISON, explanatory of the Phenomena of the Freezing Caves of Ilitzkaia Zatcheta. Notices of the Geological Soc. of London. Vol. ? p. 697.
1842. On some Phenomena observed on Glaciers, and on the Internal Temperature of large Masses of Ice or Snow, with some Remarks on the natural Ice Caves which occur below the Limit of Perpetual Snow. Notices of the Geological Soc. Vol. ? p. 699.
1843. An Account of the great Explosion for the Removal of the Round Down Cliff at Dover, in a Letter to the Editor of the Athenæum. Ath., Feb. 1843.
1843. First Notice of the Great Comet of 1843. Times Newspaper, March 20th or 21st, 1843.
1843. A second Letter to the Editor of the "Times" on do. do., March, 1843.
1843. A Letter to the Editor of the Morning Herald on do. do., March 25th or 26th, 1843.
1843. Letter to the Editor of the "Times" on the Comet and the Zodiacal Light, April, 1843.
1843. Account of an Aurora Borealis. Athenæum, No. 811, p. 465.
1843. Notice of a remarkable Photographic Process by which Dormant Pictures are rendered susceptible of Development by the Breath, or by keeping in a moist Atmosphere. Proceedings of the British Association for August, 1843.
1843. On the further Increase of η Argus. Notices of the Royal Astron. Soc., Nov. 10th, 1843.
1843. Report of the Committee consisting of Sir JOHN HERSCHEL, the MASTER OF TRINITY COLLEGE, Cambridge, the DEAN OF ELY, DR. LLOYD, and Colonel SABINE, appointed to conduct the Co-operation of the British Association in the System of Simultaneous Magnetical and Meteorological Observations. Report of the British Association for the Advancement of Science for 1843.
1843. A letter to the Editor of the Philosophical Magazine and Journal, dated Jan. 11th, 1843, On the Action of the Rays of the Solar Spectrum on the Daguerreotype Plate. From the London, Edinburgh, and Dublin Phil. Magazine and Journal of Science for Feb., 1843.
1843. Report of the Committee appointed for the Reduction of Meteorological Observations. From the Report of the British Association for the Advancement of Science for 1843.
1843. A Letter addressed to S. HUNTER CHRISTIE, Esq., Sec. Royal Soc., dated Nov. 15, 1842, on

- certain Improvements on Photographic Processes and on Parathermic Rays of the Solar Spectrum. Phil. Trans., Part I. for 1843.
1844. Astron. Nachr., No. 493. Notice of Mr. CLERHEWS's Observations of the Bifurcation of the tail of the Great Comet of 1843.
1844. Further Remarks on the Revision of the Constellations. Notices of Royal Astron. Soc., Vol. VI. p. 60.
1844. Contributions to Actinometry; also On the Amphitype, a new Photographic Process. British Association Report for 1844.
1844. Astron. Nachr., No. 520. "On M. HOUZEAU's Speculations on the Effect of Aberration in altering the Apparent Periods and Orbits of Double Stars."
1845. Address to the British Association on its Opening at Cambridge on June 19th, 1845. Report of British Association for 1845.
1845. On a Case of Superficial Color presented by a homogeneous Liquid internally colorless. Read Feb. 13th, 1845. Phil. Trans., Part I.
1845. On the Epipôle Dispersion of Light, being a Supplement to a Paper entitled "On a Case of Superficial Color presented by a homogeneous Liquid internally colorless." Read April 3d, 1845. — Phil. Trans., Part I.
1845. Sixth Report of the Committee, consisting of Sir J. HERSCHEL, the MASTER OF TRINITY COLLEGE, Cambridge, the DEAN OF ELY, Dr. LLOYD, and Colonel SABINE, appointed to conduct the Co-operation of the British Association in the System of Simultaneous Magnetical and Meteorological Observations. Report of the British Association for the Advancement of Science for 1844.
1845. Memoir of FRANCIS BAILY, Esq. From the Monthly Notices of the Royal Astronomical Society, Vol. VI, November, 1844.
1845. Report of a Committee, consisting of Sir JOHN HERSCHEL, Mr. WHEWELL, and Mr. BAILY (deceased), appointed by the British Association in 1840, for revising the Nomenclature of the Stars. Report of the British Association for the advancement of Science for 1844.
1845. "On a Seed-depositing Dibble," in a letter to the Editor of the Gardener's Chronicle. G. C. 1845, No. 12.
1846. Astron. Nachr., No. 584. Observations of BIELA's double Comet on Jan. 28th, 1846.
1846. Notices of Astron. Soc., Vol. VII. p. 95. Letter to Captain SMYTHE on the change of Magnitude of β Ursæ Minoris.
1846. Letter to Editor on LEVERRIER's (ADAM's) Planet. Athenæum, No. 988, p. 1019.
1846. Letter to Editor of the Guardian on do. do. Guardian, Oct. 28th, 1846.
1846. Letter to Editor of the Times, Oct., 1846.
1847. Obituary Notice of BESSEL. Notices of Astron. Soc., Feb. 12th, 1847.
1847. On the Action of Lime in Preserving liquid Manure. Gardener's Chronicle, 1847, No. 15, p. 244.
1847. Results of Astronomical Observations made during the Years 1834, 5, 6, 7, 8, at the Cape of Good Hope, being the Completion of a Telescopic Survey of the whole Surface of the visible Heavens, commenced in 1825. SMITH & ELDER, London.
1847. Astron. Nachr., No. 616. Extract of a Letter to Prof. SCHUMACHER on the Nomenclature of the small Planets.
1847. Letter to Editor of the Times on the Planet Neptune.
1847. Review of HUMBOLDT's Kosmos. Edinburgh Review, Jan. 1848.
1848. Obituary Notice of Miss CAROLINE LUCRETIA HERSCHEL. Athenæum, No. 1066.
1848. Address to the Royal Astron. Soc. on delivering Testimonials of Merit to Messrs. HANSEN, HENCKE, HIND, BISHOP, LUBBOCK, LEVERRIER, ADAMS, ARGELANDER, AIRY, EVEREST, and WEISSE. Notices, Feb., 1848.
1848. Athenæum, No. 1100, p. 1176. Letter to Editor on the Planet Neptune.
1848. Athenæum, No. 1100. On dividing the Vanishing Lines in Perspective.
1848. Athenæum, No. 1099, p. 1149. Account of a Lunar Rainbow.
1848. On the Effects of the Moon on the Weather. (Lord LOVELACE's Essay on Climate as connected with Husbandry.)
1849. Address delivered on presenting the Honorary Medal of the Royal Astronomical Society to WILLIAM LASSELL, Esq., of Liverpool. Royal Astron. Soc., Vol. IX. No. 4.
1849. On the Determination of the most probable Orbit of a Binary Star. Read April 13th, 1849. Trans. Royal Astron. Soc.
1849. Outlines of Astronomy. LONGMAN, London. (This has been translated into Arabic and Chinese.) N. B.—A 5th Edition much enlarged, published in 1858.
1849. Preface, Article Meteorology, Appendix to Astronomy, and Appendix to Ethnology in the "Naval Officer's Scientific Manual," edited by Sir J. HERSCHEL.
1850. Athenæum for Sat., Feb. 2d, 1850. Latin Epigram—"In cupidinem flantem." Also, Notice of an Error of 2 days left uncorrected in the Gregorian Reformation of the Calendar.
1850. Review of M. QUETELET's Lettres sur la Théorie des Probabilités, and Translation. Edinburgh Review, July, 1850.

1850. On the Algebraic Expression of the Number of Partitions of which a given Number is susceptible. Read May 16th, 1850. Phil. Trans., Part II., for 1850.
1851. Reclamation on Mr KEMP's claiming my Process for Separation of Iron from other Metals. Phil. Magazine, No. 170.
1853. On the Substitution of Bromine for Iodine in Photography. Journal of the Photographic Soc., No. 6, p. 70, June 21st, 1853.
1853. Athenæum, No. 1341, p. 83. Letter to Editor on Mr. J. STEWART's new Photographic Processes.
1854. Evidence on Examination before the Select Committee of the House of Commons on the Decimal Coinage. See Report of Committee, 1853, p. 45-62.
1854. Circular issued to Bankers by Sir J. H., Master of the Mint, proposing Inquiries respecting the Circulating of the Florin. Report of the Committee for Decimal Coinage, 1857, p. 125.
1854. Analysis of Replies of Bankers, &c., to the Queries on the Circulation of the Florin. Preliminary Report of Committee on Decimal Coinage, 1857.
1856. Remarks on Slaty Cleavage and the Contorsion of Rocks, in a Letter to T. TYNDALE, Esq., F. R. S.
1856. Letter to Dr. BOOTH, Chairman to the Society of Arts, &c., on the proposed Examination System of the Society. Literarium and Educational Gazette, March 26th, 1856.
1857. The article "Meteorology," Encyclopædia Britannica, Vol. XIV.
1857. Letter in the "Times" signed "Helioscopus," April, 1857.
1857. A volume entitled "Essays from the Edinburgh and Quarterly Reviews with Addresses and other Pieces." LONGMAN & Co., London, one vol. 8vo.
1857. Replies to Lord OVERSTONE's Queries on the Decimal Coinage. Printed in the Report of the Committee for considering the Subject of the Decimal Coinage, 1857.
1858. Sensorial Vision: a Paper read before the Philosophical and Literary Society of Leeds, Sept. 30th, 1858.
1858. Introductory Address to the Chemical Section of the British Association on taking the Chair of that Section in Leeds, in 1858. Sept. 23d.
1858. Speech in the Town Hall at Leeds, on Occasion of presenting the Prizes for the Oxford Competitive Examination to the Successful Candidates. Reported in the Leeds Mercury, Sept. 1858.
1858. Report of the Joint Committee of the Royal Soc., and the British Association for procuring Continuance of the Magnetic and Meteorological Observations.
1858. Speech in the Leeds Commercial Exchange, on the Occasion of presenting Prizes to the Successful Candidates of the Young Men's Christian Institute for the Society of Arts Competitive Examinations.
1858. A Letter to the Editor of the Photographic News, dated Oct. 31st, 1858. Photographic News.
1858. A Letter to the Editor of the Photographic News, dated Nov. 6th, 1858, on the "Stereoscopic Angle."
1858. Obituary Notice of the late Very Rev. Dr. PEACOCK, DEAN OF ELY. From the Proceedings of the Royal Soc. for Nov 30, 1858.
1859. Remarks on Color Blindness. From the Proceedings of the Royal Soc. for May 26th, 1859.
1859. Manual of Meteorology. Extracted from the Admiralty Manual of Scientific Enquiry. Third Edition, 1859.
1859. A Letter to the Chairman of the Musical Pitch Committee on Uniform Musical Pitch, dated June 14th, 1859.
1859. The article "Physical Geography," Encyclopædia Britannica, Vol. XVII.
1859. A Letter to the Editor of the Photographic News, dated August 20th, on the Action of the Solar Spectrum upon certain Compounds of Silver. Photographic News.
1859. On a New Projection of the Sphere. Communicated to the Royal Geographical Society, March, 1859. Read April, 1859. Transactions of R. G. S., 1860-61.
1859. A Letter to the Chairman of the Society of Arts Committee for considering and reporting on the Adoption of a Standard of Musical Pitch. Journal of the Society of Arts, No. 346, Vol. VII., July 8th, 1859.
1859. A third Letter to the Editor of the Photographic News, dated July 16, 17, on the Action of the Solar Spectrum on Iodized, Bromized, and other Papers. Photographic News, July 17th, 1859.
1859. A fourth Letter, dated Aug. 20th, to the Editor of the Photographic News, on the Use of Lactate of Silver, and on the Spectra impressed on Iodinated, Brominated, and other Papers, by the Action of the Solar Rays. Photographic News, Sept. 9th, 1859, Vol. III. p. 2.
1860. Athenæum, No. 1696, p. 581. British Modular Standard of Length. — Letter to Editor.
1860. Demonstration of Dr. BRINKLEY's Expression of the Coefficients in the Development of an Exponential Function. Phil. Trans. Royal Soc. London, 1860.
1860. Athenæum, No. 1697, p. 216. A second Letter to the Editor of Athenæum on the Standard of Length.
1860. A fifth letter to Editor of Photographic News on Representation of Scenes in Action, and On Instantaneous Photography.

1860. Replies to Queries circulated by the Royal Commission for Light-House Buoys and Beacons, printed in the Report of the Commission.
1860. A Letter to the Editor of the "Photographic News," dated June 30th, 1860.
1861. An Article on a Swing Cot for Sea Voyages. From the Journal of the Society of Arts, Jan. 4th, 1861.
1861. The article "Telescope," Encyclopædia Britannica, Vol. XXI.
—? Notice of a remarkable Occurrence of Serpentine at the Junction of Sienite with the Dolomite of the Tyrol.

This Catalogue of Titles was prepared by the author at our request. It would have been very difficult, if not impossible, for any other hand to have made it so complete. It is an interesting and valuable record of the incessant labor of half a century: remarkable both for the number of papers included and the variety of subjects treated. We had intended to give in this connection a short biographical sketch of the distinguished author, but want of space forbids. We will, however, refer our readers to a *Biographical Dictionary*, now in course of publication by MESSRS. GRIFFIN of London, for a sketch, which is, so far as it goes, exact, and has received SIR JOHN HERSCHEL's sanction.

While expressing our sincere thanks for the preparation of this catalogue and for a large number of the memoirs included in it, we would also acknowledge the receipt of the following very valuable papers by the author's illustrious father, SIR WILLIAM HERSCHEL: —

1789. Catalogue of a Second Thousand of New Nebulæ and Clusters of Stars; with a few Introductory Remarks on the Construction of the Heavens. Phil. Trans. Read at the Royal Society, June 11th, 1789.
1796. On the Method of observing the Changes that happen to the Fixed Stars; with some Remarks on the Stability of the Light of our Sun. To which is added a Catalogue of Comparative Brightness, for ascertaining the Permanency of the Lustre of Stars. Read before the Royal Society, Feb. 25th, 1796. Phil. Trans.
1796. On the Periodical Star α Herculis; with Remarks tending to establish the Rotatory Motion of the Stars on their Axes. To which is added a Second Catalogue of the Comparative Brightness of the Stars. Read before the Royal Society, June 9th, 1796. Phil. Trans.
1797. A Third Catalogue of the Comparative Brightness of the Stars; with an Introductory Account of an Index to Mr. FLAMSTEED's Observations on the Fixed Stars, contained in the Second Volume of the *Historia Cœlestis*. To which are added several useful Results derived from that Index. Read before the Royal Society, May 18th. Phil. Trans.
1799. A Fourth Catalogue of the Comparative Brightness of the Stars. Read before the Royal Society, February 21st. Phil. Trans.
1802. Catalogue of Five Hundred New Nebulæ, Nebulous Stars, Planetary Nebulæ, and Clusters of Stars; with Remarks on the Construction of the Heavens. Read before the Royal Society, July 1st, 1802. Phil. Trans.
1803. Account of the Changes that have happened, during the last twenty-five years, in the Relative Situation of Double Stars; and an Investigation of the Cause to which they are owing. Read before the Royal Society, June 9th. Phil. Trans.

Editorial Items.

SIR JOHN HERSCHEL'S PORTRAIT. — It gives us sincere pleasure to be able to present to our readers an excellent likeness of Sir JOHN HERSCHEL. It was engraved by the distinguished artist, H. WRIGHT SMITH, Esq. of Boston, from a Photograph in the possession of Miss MARIA MITCHELL, of Nantucket, who says the likeness is as good as it can be. Miss MITCHELL has kindly allowed us the perusal of an extract of a letter from Lady HERSCHEL, in which a decided preference is given to this Photograph.

We can assure our friends that the artist has been remarkably successful in reproducing the likeness; and it is with the deepest satisfaction that we present it to the numerous admirers of the distinguished original on this side of the Atlantic, by whom we feel assured it will be received with the warmest welcome.

The following gentlemen have sent solutions of the problems in the December number:—

H. J. MCMURPHY, Londonderry, N. H.; ASHER B. EVANS, Madison University, Hamilton, N. Y.; W. L. MARCY, Coventry, N. Y.; J. M. TERRILL, Linneus, Me.; G. B. HICKS, Cleveland, Ohio; EDWIN A. DODDS, Gouverneur, N. Y.; WILLIAM TIMPSON, White Plains, N. Y.; JAMES CLARK, Wayne, Maine; E. S. STEARNS, Chester, N. J.; DAVID TROWBRIDGE, Perry City, N. Y., W. W. LEMMON, McKendree College, Lebanon, Ill.; W. G. N., Philadelphia, Pa.; H. P. HENDERSON, N. Y. City.

PRIZES FOR SOLUTIONS. — Those entitled to these Prizes will please address Messrs. SEVER & FRANCIS, stating the number of the Monthly in which the prize is awarded, its amount, the books desired, and how they should be sent. It must not be forgotten that the postage on books must be *prepaid*.

It would add very much to the interest and value of this department of the Monthly if the competition could be largely increased. It is not certain that the good accomplished at present warrants the expense, labor, and space in the Monthly devoted to these prizes.

BOOKS RECEIVED. — A Treatise on the Differential and Integral Calculus. By A. E. CHURCH, LL. D., Professor of Mathematics in the U. S. Military Academy, West Point. A new edition. Published by A. S. Barnes & Burr, 51 and 53 John Street, N. Y. (Notice in next number.)

An Elementary Treatise on the Dynamics of a System of Rigid Bodies. With numerous Examples. By EDWARD JOHN ROUTH, M. A., Fellow and Assistant Tutor of St. Peter's College, Cambridge, Examiner in the University of London. Cambridge: Macmillan & Co.; and 23 Henrietta Street, Covent Garden, London. 1860.

THE
MATHEMATICAL MONTHLY.

Vol. III . . . MAY, 1861. . . No. VIII.

PRIZE PROBLEMS FOR STUDENTS.*

I. Let $ABCD$ be a rectangular parallelogram, G, H , two points in AD and DC , and E, F , two points in AB and BC respectively, such that GH equals EF . Join AH, AF, CG, CE , and show that $AH^2 + CG^2 = AF^2 + CE^2$.

II. ABC is an isosceles triangle, AB the base, D a point in a straight line parallel to AB , DE perpendicular to AB ; prove that the difference of the triangles ADC, AEC , is a constant area.

III. If in the continued fraction

$$\sqrt{n} = a + \frac{1}{b + \frac{1}{2c + \frac{1}{b + \sqrt{n}}}}$$

we put $a + \frac{1}{b} = M$, $a + \frac{1}{b + \frac{1}{c}} = N$, then $MN = n$.

IV. Let

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad lx + my + nz = 0,$$

be the equations of an ellipse in space; prove that the principal axes of this ellipse are the roots of the equation

$$\frac{l^2 a^2}{z^2 - a^2} + \frac{m^2 b^2}{z^2 - b^2} + \frac{n^2 c^2}{z^2 - c^2} = 0.$$

* The right-hand member of equation 1, Prob. I., March No., should be y^2 , instead of $4y^2$.

V. One of two homofocal ellipses is inscribed in a triangle, and the other circumscribed about it; show that the following relation exists between the semi-transverse axes and eccentricity

$$e^8 - 4e^6a^2 + 6e^4a^2a'^2 - 4e^2a^2a'^4 + a^4a'^4 = 0.$$

Solutions of these problems must be received by July 1, 1861.



AWARD OF THE PRIZES FOR SOLUTIONS OF PROBLEMS
IN No. IV., Vol. III.*

THE first prize is awarded conjointly to G. B. HICKS, Cleveland, Ohio, and DAVID TROWBRIDGE, Perry City, N. Y., their solutions being considered of equal merit.

The second prize is awarded to JOHN Q. HOLLISTER, Hamilton College, Clinton, N. Y.

The third prize is awarded to GEORGE W. PIERCE, Freshman Class, Harvard College.

The fourth prize is awarded to MARCUS M. RHOADES, Mount Pleasant College, Huntsville, Mo.

* To prevent misapprehension, we will state here, that it was never intended that any competitor should draw the same prize twice during the same year, although credit will be given him as being otherwise entitled to it; and no prize will be awarded for solutions which do not in themselves possess sufficient merit to fairly entitle them to it. It is a matter of extreme satisfaction that a few of the competitors always send excellent, and often elegant solutions of all the problems, and are certainly preparing themselves for important and responsible positions demanding mathematical attainments and ability. We sincerely hope to witness a great increase in competition for these prizes, and to this end we now remove the restriction limiting the competition to students in the United States, and invite all students interested in this department of the Monthly to send us their solutions. To the inquiry of our young English friends, we reply, that we shall be most happy to have them compete for and carry off these prizes. Your letters sent to us, *post-paid*, at *Cambridge, Mass., U. S. A.*, will be pretty sure to be promptly received. — ED.

PRIZE SOLUTION OF PROBLEM I.

By W. W. LEMMON, M'Kendree College, Lebanon, Ill.

$ABCD$ is a square, AC its diagonal; bisect AD in E , join BE cutting AC in F ; then will

$$\frac{\triangle AEF}{1} = \frac{\triangle CEF}{2} = \frac{\triangle ABE}{3} = \frac{\triangle BCF}{4}.$$

Bisect AB in G and join FG . It is evident that

$$AEF = AFG = GFB. \therefore AEF = \frac{1}{2} AFB = \frac{1}{4} AEB.$$

But $AEC = AEB$; subtract AEF from each, and $CEF = AFB$. The triangles AEF , $BF C$ are similar; therefore

$$\frac{AEF}{BFC} = \frac{AE^2}{BC^2} = \frac{1}{4}. \therefore AEF = \frac{1}{4} BFC.$$

PRIZE SOLUTION OF PROBLEM II.

BY SEVERAL OF THE COMPETITORS.

If $\frac{a-b}{ay+bx} = \frac{b-c}{bz+cx} = \frac{c-a}{cy+az} = \frac{a+b+c}{ax+by+cz}$, then each of these fractions $= \frac{1}{x+y+z}$.

Since these ratios are equal, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents; therefore either fraction equals the sum of all the numerators *divided by the sum of all the denominators*,

$$\therefore \frac{a+b+c}{(a+b+c)(x+y+z)} = \frac{1}{x+y+z}.$$

PRIZE SOLUTION OF PROBLEM III.

If d be the perpendicular, dropped from the angle C on the opposite side of a triangle, show that $\sin C = \frac{cd}{ab}$. Deduce the formula

$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

From the vertex B drop a perpendicular d' on the opposite side. Then $a \sin C = d'$, and calling the area of ABC , A , we get

$$ab \sin C = bd' = 2A = cd; \therefore \sin C = \frac{cd}{ab}.$$

Now, $c = a \cos B + b \cos A$, $d = b \sin A = a \sin B$;

$$\therefore \frac{cd}{ab} = \sin A \cos B + \cos A \sin B = \sin C = \sin(A + B),$$

since $A + B + C = 180^\circ$. This is substantially the solution given by all the competitors.

PRIZE SOLUTION OF PROBLEM IV.

An hyperbola, having its asymptotes parallel to the axes of co-ordinates, may be made to pass through the angular points of the quadrilateral formed by the four lines whose equations are

$$\begin{array}{ll} (1) \ y + x + 6a = 0, & (3) \ y + 6x + a = 0, \\ (2) \ y + 2x + 3a = 0, & (4) \ y + 3x + 2a = 0. \end{array}$$

Denote the equations (1), (2), (3), (4) by $\alpha, \beta, \gamma, \delta$; then

$$(1) \quad \alpha \gamma + k \beta \delta = 0$$

is the equation of a conic passing through the angular points of the quadrilateral $(\alpha \beta \gamma \delta)$; for it is satisfied by $\alpha = 0, \beta = 0$; $\alpha = 0, \delta = 0$; $\gamma = 0, \beta = 0$; $\gamma = 0, \delta = 0$; k being an arbitrary constant. We have then

$$(2) \ y^2 + 7xy + 6x^2 + 7ay + 37ax + 6a^2 + k(y^2 + 5xy + 6x^2 + 5ay + 13ax + 6a^2) = 0.$$

To remove the second powers of x and y make $k = -1$,

$$(3) \quad \therefore xy + ay + 12ax = 0.$$

Put $x = x' + b, y = y' + c$, then (3) becomes

$$(4) \ x'y' + (c + 12a)x' + (a + b)y' + bc + ac + 12ab = 0;$$

and by making $c = -12a, b = -a$, we get $x'y' = 12a^2$, the well-known equation of the hyperbola referred to its asymptotes. Therefore (3), which passes through the angular points of the quadrilateral, is an hyperbola having its axis of co-ordinates parallel to its asymptotes. This solution was given by Messrs. HICKS and TROWBRIDGE.

MR. ASHER B. EVANS gave the following solution :—

Suppose the centre of the hyperbola to be at the point (θ, φ) , and remove the origin of coördinates to this point ; then will equations, (1), (2), (3), (4) become

$$\begin{aligned} (5) \quad y + x + 6a - \theta - \varphi &= 0, & (7) \quad y + 6x + a - 6\theta - \varphi &= 0, \\ (6) \quad y + 2x + 3a - 2\theta - \varphi &= 0, & (8) \quad y + 3x + 2a - 3\theta - \varphi &= 0. \end{aligned}$$

Let A, B, C, D represent the intersection of (1) and (2), (1) and (4), (2) and (3), (3) and (4), in forming the quadrilateral $ABCD$. From equations (5), (6), (7), (8), we readily find

$$\begin{aligned} A &= (\theta + 3a, \varphi - 9a), & B &= (\theta + 2a, \varphi - 8a), \\ C &= (\theta + \frac{1}{2}a, \varphi - 4a), & D &= (\theta + \frac{1}{3}a, \varphi - 3a). \end{aligned}$$

The equation to the hyperbola, referred to its asymptotes as axes, is

$$(9) \quad xy = c^2.$$

If the hyperbola pass through A, B, C , we must have

$$\begin{aligned} (10) \quad xy = c^2 &= (\theta + 3a)(\varphi - 9a) = (\theta + 2a)(\varphi - 8a) \\ &= (\theta + \frac{1}{2}a)(\varphi - 4a). \end{aligned}$$

Equation (10) gives $\theta = a, \varphi = 12a$; hence

$$\begin{aligned} (11) \quad xy = c^2 &= (\theta + 3a)(\varphi - 9a) = (\theta + 2a)(\varphi - 8a) \\ &= (\theta + \frac{1}{2}a)(\varphi - 4a) = (\theta + \frac{1}{3}a)(\varphi - 3a) = 12a^2. \end{aligned}$$

Since equation (11) exists, the hyperbola which passes through A, B, C , also passes through D , and hence through all the angular points of the quadrilateral $ABCD$.

PRIZE SOLUTION OF PROBLEM V.

By DAVID TROWBRIDGE, Perry City, N. Y.

Three lines, x, y, z , are drawn from the angular points of a triangle to a point within it. Show that when $x^n + y^n + z^n$ is a minimum we have

$$\frac{\sin(x, y)}{x^{n-1}} = \frac{\sin(x, z)}{y^{n-1}} = \frac{\sin(y, z)}{x^{n-1}},$$

in which (x, y) denotes the angle between x and y .

Let ABC be the triangle, and O the point within it, $OC = x$, $OB = y$, $OA = z$, and the angle $(x, a) = \theta$. Then we readily get

$$\begin{aligned} u &= x^n + y^n + z^n, \\ &= x^n + (a^2 + x^2 - 2ax \cos \theta)^{\frac{n}{2}} + (b^2 + x^2 - 2bx \cos (C - \theta))^{\frac{n}{2}}, \\ D_x u &= nx^{n-1} + ny^{n-2}(x - a \cos \theta) + nz^{n-2}(x - b \cos (C - \theta)) = 0, \\ (1) \quad &= x^{n-1} + xy^{n-2} + xz^{n-2} - ay^{n-2} \cos \theta - bz^{n-2} \cos (C - \theta) = 0, \\ D_\theta u &= nax y^{n-2} \sin \theta - nbx z^{n-2} \sin (C - \theta) = 0, \\ (2) \quad &= ay^{n-2} \sin \theta - bz^{n-2} \sin (C - \theta) = 0. \end{aligned}$$

By putting $B = x^{n-2} + y^{n-2} + z^{n-2}$, (1) becomes

$$(3) \quad Bx = ay^{n-2} \cos \theta + bz^{n-2} \cos (C - \theta).$$

Multiplying (3) by $\sin \theta$, and (2) by $-\cos \theta$, and adding the results, we get

$$Bx \sin \theta = bz^{n-2} \sin C, \quad \text{or} \quad Bax \sin \theta = abz^{n-2} \sin C.$$

But $ax \sin \theta = xy \sin (x, y) =$ twice the area of BOC ;

$$(4) \quad \therefore \frac{Bxyz \sin (x, y)}{x^{n-1}} = ab \sin C = \text{twice the area of the triangle } ABC.$$

Similarly,

$$\frac{Bxyz \sin (x, z)}{y^{n-1}} = ac \sin B, \quad \frac{Bxyz \sin (y, z)}{x^{n-1}} = bc \sin A;$$

$$\therefore \frac{\sin (x, y)}{x^{n-1}} = \frac{\sin (x, z)}{y^{n-1}} = \frac{\sin (y, z)}{x^{n-1}}.$$

The condition of a minimum

$$D_x^2 u \cdot D_\theta^2 u - (D_{x, \theta}^2 u)^2 > 0.$$

is readily computed.

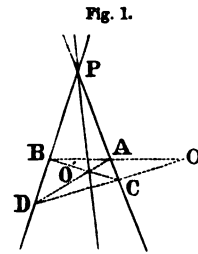
Mr. G. B. HICKS computed the values of x, y, z in a very elegant manner for $n = 1$, and $n = 2$, which we reluctantly omit for want of space.

ON THE GEOMETRICAL CONSTRUCTION OF CERTAIN
CURVES BY POINTS.

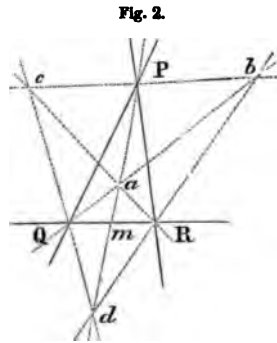
By H. A. NEWTON, Professor of Mathematics in Yale College.

1. DEFINITION. — *The polar to a point with respect to an angle* is that straight line which, together with the line drawn from the point to the vertex of the angle, divides the angle harmonically. (*Géom. Sup.*, Art. 347.)*

2. It may be constructed by the straight edge alone. For let O (Fig. 1) be the given point, and APB the given angle. Draw OAB and OCD , any two straight lines, through O to cut PA in A and C , and PB in B and D ; draw AD and BC to cut each other in O' , and draw PO' . The line PO' will be the polar of the point O with respect to the angle APB . (*Géom. Sup.*, Art. 346.)



3. THEOREM. — Let PQR (Fig. 2) be any triangle, and a any point in the same plane. Draw aP , aQ ,



aR , and also Pb , Qc , and Rd , the polars to a with respect to the three angles of the triangle PQR . Then the lines Pa , Qc , and Rd will meet in one point, d , the lines Pb , Qa , and Rd will meet in one point, b , and Pb , Qc , and Ra will meet in one point, c .

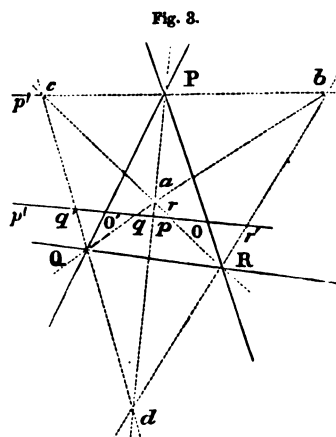
For let Pa meet QR , Qc , and Rd in m , d' , and d respectively. The segment Pm is cut harmonically by Qa and Qd' , and also by Ra and Rd . Therefore d and d' coincide; or, in other words, Pa , Qc , and Rd meet in one point. In like manner the other lines may

* The references are to the *Traité de Géométrie Supérieure*, par M. CHASLES, 8vo, Paris, 1852, Bachelier. We cannot too highly recommend this work of the great French Geometer.

be proved to meet in b , and in c . This proposition is proved also in the *Géom. Sup.*, Art. 373.

4. In the figure thus constructed the six lines joining the four points a, b, c, d may be considered the four sides and two diagonals, or in other words the six sides of a complete quadrilateral, $abcd$. The points P, Q , and R will be the points of intersection of the three couples of opposite sides.

5. THEOREM. — The three polars of any point, O (Fig. 3), with respect to the three angles formed at P, Q , and R by the opposite sides of the quadrilateral meet in one point, O' .



This proposition is demonstrated by CHASLES (*Géom. Sup.*, Art. 347), as follows: Let the polars of O , with respect to the angles aPb and aQc , meet in O' ; it is to be proved that the line joining O' and R is polar to O , with respect to the angle aRd . Draw OO' , and let this line meet the sides of the quadrilateral in $p, p'; q, q'; r, r'$. These six points are in involution (*Géom. Sup.*, Art. 339); and since the segment OO' divides harmonically the

segments pp' and qq' , the points O and O' are the double points of the involution (*Géom. Sup.*, Art. 205). Hence the segment rr' divides the segment OO' harmonically, and O' is on the polar of O with respect to the angle aRd .

6. Since there is but one polar to a point with respect to an angle, it is evident that, having constructed the quadrilateral (or having assumed the point a), there is *in general* to one point, O , of the plane PQR one point, O' , and only one, where the polars of O intersect.

These two points, O and O' , have important relations to each other; and in what follows, each will be said to be *conjugate* to the other.

7. If O is on one of the sides of the quadrilateral, as ad , then,

1st, O' is on the same line ad ; and

2d, the segment, OO' , divides ad harmonically; for,

1st, the polar of O , with respect to the angle aPb , is the side of the angle Pa ; and

2d, the lines OR and $O'R$ must divide the angle aRd , and hence the segment ad , harmonically.

8. A fasceau of conics circumscribing the quadrilateral $adQR$ will, by a well-known theorem, cut the line bc in a series of couples of points in involution. And since the two lines ba , bd may be considered as one of these conics, the point b is one of the double points of the involution. The point c , for a like reason, is the other double point. Therefore each segment, as mn , intercepted on the line bc by a conic about $adQR$, is divided harmonically by b and c (*Géom. Sup.*, Art. 205). Therefore m and n are points conjugate to each other (Art. 7). Hence, if a conic be described to pass through a and d , two summits of the quadrilateral, and through Q and R , those two angles of the triangle not on the line ad , then will m and n , the two points where this conic cuts the opposite side, bc , of the quadrilateral, be conjugate to each other.

9. The point a is entirely arbitrary. The positions of conjugate points may, however, be more readily conceived if a is taken the centre of the circle inscribed in the triangle PQR . Then will the lines aP , aQ , aR bisect the angles of the triangle, the lines Pb , Qc , and Rd will bisect the exterior angles of the triangle, and be perpendicular respectively to the lines aP , aQ , and aR . The points b , c , and d will be the centres of the circles touching the sides of the triangle PQR , and the angles aPO , aQO , and aRO will be equal respectively to $aP'O$, $aQ'O$, and $aR'O$.

The centre of a circle inscribed in a given triangle cannot, however, be determined by a straight edge alone; and hence, in con-

structions where the compasses are not to be used, the point a must not be subject to the above condition.

10. DEFINITIONS.—If the point O move along any curve, its conjugate point O' will describe a second curve. This second curve will be called *the conjugate* of the first curve. The first curve is evidently the conjugate of the second.

The points P , Q , and R will be called *base points*. The three lines PQ , PR , and QR will be called *axes*.

11. If two curves intersect in n points, none of which are base points, the two conjugate curves will evidently intersect in n points, none of which are base points.

12. If two curves touch each other at a point not a base point, the two conjugate curves will also touch each other.

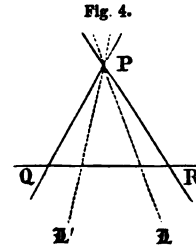
For two curves, $\mathfrak{A}$ and $\mathfrak{B}$, are tangent to each other in a point O , when the segment intercepted by them on any transversal indefinitely near to O is an infinitesimal of the second order. Let the straight line Pmn , indefinitely near to O , cut $\mathfrak{A}$ and $\mathfrak{B}$ in m and n , and let m' , n' , and O' be the points conjugate to m , n , and O , and therefore on the curves conjugate to $\mathfrak{A}$ and $\mathfrak{B}$. The points, m' , n' , and P will evidently be in a straight line, which will be indefinitely near to the line $O'P$. The angle made by the lines mQ and nQ , and hence the angle made by $m'Q$ and $n'Q$ will be infinitesimals of the second order. Therefore, if the directions of $m'P$ and $m'Q$ make a finite angle, the segment $m'n'$ is an infinitesimal of the second order, and the curves conjugate to $\mathfrak{A}$ and $\mathfrak{B}$ are tangent to each other at O' .

13. When the point O' is on an axis, as PQ , or is at infinity, this demonstration fails. In the first case, the point O is the base point R . It is easy to see that if two curves intersect in an axis, their conjugate curves touch each other in the opposite base point. This is only an apparent exception to the proposition of Art. 11th, for

the two curves tangent at the base point may be considered as intersecting each other in the base point and in a second point indefinitely near to the base point.

14. The curve conjugate to a straight line, $\mathfrak{L}$, passing through one, and only one, of the base points, as P , is a straight line passing through the same base point, P .

For let $\mathfrak{L}$ (Fig. 4) be drawn so that $\mathfrak{L}$ and $\mathfrak{L}'$ shall divide harmonically the angle made at P by the opposite sides of the quadrilateral. Then if O moves along $\mathfrak{L}$, the point O' must move along $\mathfrak{L}'$; that is, if $\mathfrak{L}$ and $\mathfrak{L}'$ are conjugate to each other.



15. If the line $\mathfrak{L}$ is an axis, there is no conjugate line.

For since PQ and PR divide harmonically the angle made at P by the opposite sides of the quadrilateral, the point conjugate to any point on PQ is on PR . For a like reason the point conjugate to any point on PQ is on QR . Hence the point of intersection, R , is to be considered the conjugate of every point in PQ .

It was stated above (Art. 6), that *in general* each point, O , of the plane, PQR , had one, and only one, conjugate point. The base points, P , Q , and R , are the only exceptions.

16. The curve conjugate to a straight line, not passing through a base point, is a conic that passes through the three base points.

For let a, b, c, d , &c., be points on the straight line, and a', b', c', d' , &c., be the points respectively conjugate to them, and let lines be drawn from P and Q to the several points. Then the couples of lines Pa, Pa' ; Pb, Pb' ; Pc, Pc' , &c., are in involution (*Géom. Sup.*, Art. 244), and hence the anharmonic function of the lines Pa', Pb', Pc' , and Pd' is equal to that of Pa, Pb, Pc , and Pd (*Géom. Sup.*, Arts. 236 and 13); which is equal to that of Qa, Qb, Qc , and Qd ; which is equal to that of Qa', Qb', Qc' , and Qd' . Therefore the locus of the points a', b', c', d' , &c., is a conic passing through P and Q (*Géom. Sup.*, Art. 542), and hence by symmetry through R .

17. Conversely, the conjugate to a conic passing through the three base points, is a straight line that does not pass through a base point.

18. DEFINITIONS. — A curve of the n th degree is one that is cut by every straight line in its plane in n points (real and imaginary), and in n points only.

A *conic* is a curve of the second degree.

A *cubic* is a curve of the third degree.

SALMON proposed that the term cubic, used in Algebra to denote an equation of the third degree, should be used in Geometry to denote a curve of the third degree. As we shall have occasion frequently to speak of curves of the fourth degree, we shall make a similar use of the term biquadratic, by which we shall mean a curve of the fourth degree.

19. The curve conjugate to a conic $\mathbb{C}$ that passes through two, and only two, of the base points, as P and Q , is a conic that passes through the same two base points, and does not pass through the third base point.

It is a conic. For any conic through P , Q , and R cuts the conic $\mathbb{C}$ in four points; that is, in two points other than the base points, P and Q . Therefore any straight line (Art. 17) cuts the conjugate to $\mathbb{C}$ in two, and only two, points (real or imaginary), namely, the two points conjugate to the intersections of the first two conics.

Moreover, the conic $\mathbb{C}$ crosses the axis PR in one point, D , other than P . The points conjugate to points indefinitely near to D will evidently be on the line through Q , conjugate to DQ , and these points will be indefinitely near to Q ; in other words, the conjugate to $\mathbb{C}$ passes through Q , and has for its tangent the line conjugate to DQ . For a like reason it passes through P . It does not, however, pass through R ; for its conjugate, that is, $\mathbb{C}$, would then be a straight line (Art. 17).

20. The curve conjugate to a conic $\mathfrak{C}$ through one, and only one, base point, P , is a cubic that passes through the other two base points, and has a double point at P .

It is a cubic. For any conic through P , Q , and R cuts $\mathfrak{C}$ in three, and only three, points (real and imaginary) other than P . Hence any straight line cuts the conjugate to $\mathfrak{C}$ in three, and only three, points.

Again, since $\mathfrak{C}$ cuts PQ and PR each in one point other than P , the conjugate to $\mathfrak{C}$ passes through the points R and Q . And since $\mathfrak{C}$ crosses QR in two points, neither of which is a base point, there are two directions in which the conjugate to $\mathfrak{C}$ passes through P ; that is, there are two tangents (real or imaginary) to the curve at P , and hence P is a double point. If the tangents are imaginary, then P is an isolated point.

21. The curves conjugate to a conic, $\mathfrak{C}$, that does not pass through any base point, is a biquadratic, having double points at each base point.

It is a biquadratic. For any conic through P , Q , and R cuts $\mathfrak{C}$ in four points (real or imaginary), neither of which is a base point. Hence any straight line cuts the conjugate to $\mathfrak{C}$ in four points (real or imaginary).

Again, $\mathfrak{C}$ cuts each axis in two points, neither of which is a base point. Hence the conjugate to $\mathfrak{C}$ has two tangents (real or imaginary) at each base point.

22. These are single examples of the following general theorem. Let $\mathfrak{A}$ be a curve of the n th order, having a multiple point of the p order at P , a multiple point of the q order at Q , and a multiple point of the r order at R , and let $\mathfrak{A}'$ be conjugate to $\mathfrak{A}$, then,

1st, $\mathfrak{A}'$ is a curve of the $(2n - p - q - r)$ th degree; and

2d, It will have at P a multiple point of the $(n - q - r)$ th order, at Q a multiple point of the $(n - p - r)$ th order, and at R a multiple point of the $(n - p - q)$ th order.

.

1st. $\mathcal{A}'$ is a curve of the $(2n - p - q - r)$ th order.

For any conic cuts $\mathcal{A}$ in $2n$ points. If the conic passes through P, Q , and R , it will cut $\mathcal{A}$ in those points respectively, p, q , and r times; and therefore in $(2n - p - q - r)$ other points. Hence, any straight line will cut $\mathcal{A}'$ in $(2n - p - q - r)$ points.

2d. Since $\mathcal{A}$ crosses QR in $(n - q - r)$ points other than Q and R , $\mathcal{A}'$ has $(n - q - r)$ branches passing through P . For a like reason $\mathcal{A}'$ has $(n - p - r)$ branches at Q , and $(n - p - q)$ branches at R .

These branches may, of course, be imaginary by couples, and when $\mathcal{A}$ does not pass through P we must, in the above theorem, make $p = 0$, and so of the rest.

23. To illustrate the above proposition, let $\mathcal{A}$ be a cubic.

(a) If $\mathcal{A}$ passes through neither base point, $\mathcal{A}'$ is a curve of the sixth degree, having triple points at P, Q , and R .

(b) If $\mathcal{A}$ passes through only one base point, P , $\mathcal{A}'$ is a curve of the fifth degree, having a triple point at P , and double points at Q and R .

(c) If $\mathcal{A}$ passes through P and Q , and not through R , $\mathcal{A}'$ is a biquadratic having double points at P and Q , and a single point at R .

(d) If $\mathcal{A}$ passes through P, Q , and R , $\mathcal{A}'$ will be a cubic passing through P, Q , and R .

(e) If $\mathcal{A}$ has a double point at P , and does not pass through Q nor R , $\mathcal{A}'$ is a biquadratic having a triple point at P , and single points at Q and R .

(f) If $\mathcal{A}$ has a double point at P and a single point at Q , $\mathcal{A}'$ is a cubic having a double point at P , and single point at Q .

(g) If $\mathcal{A}$ has a double point at P , and passes through Q and R , $\mathcal{A}'$ is a conic passing through P , but not through Q nor R .

24. If it be required to construct by points a curve that shall satisfy certain conditions, it may be frequently accomplished by con-

structing points of a conjugate curve that shall satisfy certain corresponding conditions, and then finding the points conjugate to those constructed. This method of construction is especially applicable when the curve is to pass through certain points, or touch certain curves.

25. Thus suppose it was required to construct other points of the conic that passes through five given points.

Take three of the given points for base points, and find the points conjugate to the remaining two points. Then draw through these a straight line, and construct the conjugates to any points on this line. They will evidently be on the conic passing through the five given points (Art. 16).

26. Again, to construct by points a cubic having given one double and six single points of the curve

Take the given double point and two of the single points as base points, and construct the points conjugate to the remaining four points. Through the double point and these four conjugate points construct by points (Art. 25) a conic. The conjugate to the points of this conic will be points of the required cubic (Art. 20).

27. To construct a biquadratic having given one triple point and eight single points.

Take the triple point and two single points as base points, and construct the points conjugate to the remaining six points. Through these construct (Art. 26) a cubic having a double point at the given triple point, and then construct points conjugate to points of this cubic. They will be on the required curve (Art. 23 *e*).

28. To construct a curve of the 5th degree having given one quadruple and 10 single points.

With the quadruple point and two single points as base points, construct the conjugate to the remaining eight points. Through these construct (Art. 27) a biquadratic having the given quadruple

point for a triple point, and construct points conjugate to points of this biquadratic. They will be on the required curve.

29. In like manner may be constructed the curve of the sixth degree having given a multiple point of the 5th order and twelve single points.

And in general by repeating the same process may be constructed a curve of the n th degree having a given multiple point of the $(n-1)$ st order, and $2n$ single points.

Moreover, since the conjugate of any point may be constructed by the straight-edge alone (Art. 2), all the preceding constructions may be performed without the aid of the compasses.

30. To construct the curve of the $2n$ th degree having given three points of the n th order, one point of the $(n-1)$ st order, and $2n$ single points.

Take the three points of the n th order as base points, and the construction depends upon that of Art. 29.

31. In like manner may we construct the curve of the $(2n+1)$ st degree having given one point of the $(n+1)$ st order, three points of the $(n-1)$ st order, and $(2n+1)$ single points.

32. The construction of a cubic that shall pass through nine points is given by CHASLES (*Comptes Rendus*, Tom. XXXVI., Séance du 30 Mai, 1853). Hence, to construct the curve of the n th degree having given a point of the $(n-2)$ d order, $(n-3)$ double points and eight single points; take the point of the $(n-2)$ d order, a double point, and a single point, as base points, and the construction is made to depend upon that of the curve of the $(n-1)$ st degree having a point of the $(n-3)$ d order, $(n-4)$ double points and eight single points; that is, on a similar problem to the one required, and of the next inferior degree. Repeating the process, the construction is made to depend at last on the construction of the cubic through nine given points.

[To be continued.]

SECOND PRIZE ESSAY. — ON CENTRAL AND DISTURBING FORCES.*

By DAVID TROWBRIDGE, Perry City, N. Y.

25. ASSUME the *elements* of the planetary orbits to be known: π = the longitude of the perihelion, Ω = the longitude of the ascending node, i = the inclination. Put φ = the true, and E = the eccentric, and M = the mean anomaly; λ = the true longitude, and l = the true latitude. All these quantities are regarded as heliocentric.

The equations of an ellipse, in terms of the eccentric anomaly, are,

$$(71) \quad x = a(\cos E - e), \quad y = b \sin E.$$

Whence we obtain

$$(72) \quad \sqrt{(x^2 + y^2)} = r = a(1 - e \cos E).$$

We also have

$$(73) \quad r = \frac{a(1 - e^2)}{1 + e \cos \varphi}.$$

Equating these two values of r , and reducing, we find

$$(74) \quad \tan \frac{1}{2} \varphi = \sqrt{\frac{1+e}{1-e}} \cdot \tan \frac{1}{2} E.$$

The differential coefficients of (71), with respect to the time, are,

$$(75) \quad D_t x = -a \sin E D_t E, \quad D_t y = b \cos E D_t E.$$

If we change the axis from which the area is reckoned (Art. 5), equation (12) gives

$$(76) \quad c = x D_t y - y D_t x.$$

Substituting in (76) from (71) and (75), we find

* This Essay is a continuation of the one in Vol. II. p. 150, of the *Mathematical Monthly*.
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$$(77) \quad c = \frac{2\pi ab}{T} = ab(1 - e \cos E) D_t E.$$

If we put $\frac{2\pi}{T} = n$, reduce, and integrate, we have

$$(78) \quad nt = M = E - e \sin E,$$

since n is the angle described in the unit of time.

To find E in terms of M , we have

$$(78) \quad E = M + e \sin E, \quad \text{and} \quad \sin E = \sin [M + e \sin E].$$

The continual substitution of this value of $\sin E$ in (79) gives the series

$$(80) \quad E = M + e \sin [M + e \sin [M + e \sin [M + \dots + e \sin [M].$$

If e is small, a few terms only need be employed.

26. From equations (1), (2), (3), (41), and (20) we have

$$(81) \quad D_t^2 x = -\frac{mx}{r^3}, \quad D_t^2 y = -\frac{my}{r^3}, \quad D_t^2 z = -\frac{mz}{r^3}, \quad D_t \varphi = \frac{c}{r^2}.$$

Before we integrate these equations it will be necessary to transform them a little. For this purpose, let ANB be the ecliptic,

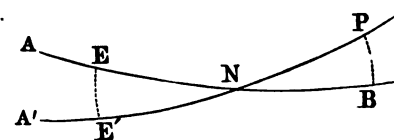
$A'NP$ the orbit of the planet, N the ascending node, A the Vernal Equinox, E' the perihelion, and P the planet. Then $AN = \mathfrak{Q}$, $AE = \pi$, and $A'N = \mathfrak{Q}'$, $A'E' = \pi'$, the orbit values of the same quantities; $ANA' = i = PNB$, $E'P = \varphi$, $AB = \lambda$, $PB = l$. Put $NP = \beta$, $NB = \beta'$. We then have, by making $E'N = \omega' = \mathfrak{Q}' - \pi'$,

$$(82) \quad \varphi = \beta + E'N = \beta + A'N - A'E' = \beta + \mathfrak{Q}' - \pi' = \beta + \omega'.$$

Differentiating with respect to t , we have

$$(83) \quad D_t \varphi = D_t \beta;$$

and hence (81) gives $D_t \beta = \frac{c}{r^2}.$



By Analytical Geometry we have

$$(84) \quad x = r \cos l \cos \beta', \quad y = r \cos l \sin \beta', \quad z = r \sin l.$$

Since PNB is a right spherical triangle, Spherical Trigonometry gives

$$(85) \quad x = r \cos \beta, \quad y = r \cos i \sin \beta, \quad z = r \sin i \sin \beta.$$

(83) and (85) being substituted in (81), give

$$(86) \quad D_t^2 x = -\frac{m}{c} \cos \beta D_t \beta, \quad D_t^2 y = -\frac{m}{c} \cos i \sin \beta D_t \beta, \\ D_t^2 z = -\frac{m}{c} \sin i \sin \beta D_t \beta.$$

Integrating, and we obtain

$$(87) \quad D_t x = C - \frac{m}{c} \sin \beta, \quad D_t y = C' + \frac{m}{c} \cos i \cos \beta, \\ D_t z = C'' + \frac{m}{c} \sin i \cos \beta.$$

To find the values of the arbitrary constants, we observe that the velocities in the direction of the co-ordinate axes are each $= 0$, when the co-ordinates are each a maximum; hence,

$$D_\beta x = D_\beta r \cos \beta - r \sin \beta = 0, \quad \text{or} \quad \frac{D_\beta r}{r} = \tan \beta.$$

$$\text{But} \quad \log r = \log a (1 - e^2) - \log (1 + e \cos (\beta + \omega'));$$

$$\text{and hence} \quad \frac{D_\beta r}{r} = \frac{e \sin (\beta + \omega')}{1 + e \cos (\beta + \omega')} = \frac{\sin \beta}{\cos \beta}.$$

Reducing,

$$\sin \beta = e [\sin (\beta + \omega') \cos \beta - \cos (\beta + \omega') \sin \beta] = e \sin \omega'.$$

This value of $\sin \beta$ in the first of (87) gives,

$$0 = C - \frac{m}{c} e \sin \omega', \quad \text{and} \quad C = \frac{m}{c} e \sin \omega'.$$

In a similar manner we can find the value of C' and C'' , and we shall thus obtain

$$(88) \quad D_i x = \frac{m}{c} (\sin \beta + e \sin \omega'), \quad D_i y = \frac{m}{c} \cos i (\cos \beta + e \cos \omega'),$$

$$D_i z = \frac{m}{c} \sin i (\cos \beta + e \cos \omega').$$

If we put $D_i x = x'$, $D_i y = y'$, $D_i z = z'$, and take the sum of the squares of (88) we find, by recollecting that $\cos(\beta + \omega') = \cos \varphi$,

$$(89) \quad D_i s^2 = v^2 = s'^2 = x'^2 + y'^2 + z'^2 = \frac{m^2}{c^2} [1 + e^2 + 2e \cos \varphi].$$

Put V_r for the velocity at the perihelion, and V_a at the aphelion, then, when $\varphi = 0$ in (89), $V_r^2 = \frac{m^2}{c^2} (1 + e)^2$, and

$$(90) \quad V_r = \frac{m}{c} (1 + e).$$

When $\varphi = 180$, $V_a^2 = \frac{m^2}{c^2} (1 - e)^2$, and

$$(91) \quad V_a = \frac{m}{c} (1 - e).$$

Whence we obtain

$$(92) \quad \frac{V_r}{V_a} = \frac{1 + e}{1 - e} = \frac{\tan^2 \frac{1}{2} \varphi}{\tan^2 \frac{1}{2} E},$$

by (74).

When the planet is at the mean distance, $\cos \varphi = -e$, and hence

$$(93) \quad V^2 = \frac{m^2}{c^2} (1 - e^2) = \frac{m}{a},$$

since $c^2 = m a (1 - e^2)$ by (51). Substituting this value of V^2 in (55), making $r = a$, and we have $h = -\frac{m}{a}$. This value of $h = C$ in (42) gives

$$(94) \quad \frac{m}{a} = \frac{2m}{r} - (x^2 + y^2 + z^2) = \frac{2m}{r} - s^2.$$

27. The planetary *elements* are dependent on the velocity with which the bodies move, or the space passed over in a given time, and this is dependent on the attractive force of the sun, which is itself a function of the distance. We may hence regard each and

every *element* as a function of the distance, velocity, and time; and if we let A represent any one of the planetary elements, we can have the equations,

$$(95) \quad A = f(x, y, z, x', y', z', t), \quad A = f(r, s', t).$$

28. All the preceding 95 equations pertain to the motion of a single planet around the sun. When this is the case, A in (95) is strictly a constant; but if we introduce a second planet, it exercises an influence that causes the value of A to vary; and the extent of this variation will depend on the amount of the disturbing force. If we imagine an ellipse to be described, passing through the position of the disturbed planet (which call P), at any instant, it is evident that by an expansion, or contraction of this ellipse, as the case may require, we can suppose it to successively represent or pass through every point that P will occupy in completing a revolution. In this case it will be seen, that, at any instant (t), the distance of P from the sun, and also its velocity, will be the same in the *variable ellipse*, as in the orbit actually described; but the force that pertains to the two orbits will be different, otherwise the orbit would still be an ellipse. *The difference of the two forces will be the disturbing force.* If A varies, it will vary with the time. If we let D' represent the differential coefficient when the function (95) pertains to an ellipse, and D the differential coefficient when it pertains to the orbit actually described, the differentiation of the first of (95) gives on both suppositions,

$$(96) \quad 0 = (D_t A) + D_x A D_t x + D_y A D_t y + D_z A D_t z + D_{x'} A D_t x' \\ + D_{y'} A D_t y' + D_{z'} A D_t z'.$$

$$(97) \quad D_t A = (D_t A) + D_x A D_t x + D_y A D_t y + D_z A D_t z + D_{x'} A D_t x' \\ + D_{y'} A D_t y' + D_{z'} A D_t z'.$$

$(D_t A)$ and $(D_t A)$ are partial differential coefficients.

According to the hypothesis, $(D_t A) = (D_t A)$, $D_t x = D_t x$, &c., $D_x A = D_x A$, &c., $D_x A = D_x A$, &c.; but $D_t x' = D_t^2 x$ = the force in the ellipse along the axis of x is not the same as $D_t x' = D_t^2 x$ = the force in the real orbit along the axis of x . The difference of (96) and (97) then gives

$$(98) \quad D_t A = D_x A (D_t^2 x - D_t^2 x) + D_y A (D_t^2 y - D_t^2 y) + D_z A (D_t^2 z - D_t^2 z).$$

If we put

$$(99) \quad Q X, Q Y, Q Z,$$

for the disturbing force along the axes, we shall evidently have

$$Q X = D_t^2 x - D_t^2 x, \quad Q Y = D_t^2 y - D_t^2 y, \quad Q Z = D_t^2 z - D_t^2 z.$$

Hence,

$$(100) \quad D_t A = Q X . D_x A + Q Y . D_y A + Q Z . D_z A.$$

This is a general formula, since it takes into account the whole amount of the disturbing force.

If we assume

$$(101) \quad A = A_x + A_y + A_z, \quad D_t A = D_t A_x + D_t A_y + D_t A_z,$$

such that

$$(102) \quad D_t A_x = Q X . D_x A, \quad D_t A_y = Q Y . D_y A, \quad D_t A_z = Q Z . D_z A,$$

we can compute the effects of the disturbing force in the direction of the co-ordinate axes (which we may regard as fixed in space) separately, and then by (101) we can conclude the joint or total effects. This, in general, will simplify the question.

If we treat the second equation of (95) in the same manner as that of the first, we shall find

$$(103) \quad D_t A = Q S . D_t' A,$$

in which

$$Q S = D_t^2 s - D_t^2 s.$$

QS acts along the tangent of the curve, and it is hence the tangential force; and (103) will serve to compute the effects of this force.

29. QX , QY and QZ can represent the sum of the disturbing forces of any number of bodies, or any one of them. If the action of all the planets had to be considered at one time, the problem of perturbation would be very difficult; but from a principle in Mechanics, “immediately flowing from the primary relations between forces and the motions which they produce, that when a number of very minute forces act at once on a system, their joint effect is the sum, or aggregate of their separate effects, at least within such limits that the original relations of the parts of the system shall not have been materially changed by their actions”; and hence it simplifies the question very considerably to consider the action of each disturbing body separately. The question is thus reduced to the famous *Problem of Three Bodies*, which at present can only be solved approximately.

30. In the Solar System, the *inclination* of the planetary orbits to each other is so small, that it introduces little error in supposing the disturbed and the disturbing body to both lie in the same plane, except when we wish to compute the effect on the nodes and inclination.

31. If we substitute mr^{-2} for R , and $-ma^{-1}$ for $C=h$, in equation (28), and integrate, we find

$$D_t r^2 = \frac{2m}{r} - \frac{m^2}{r^2} - \frac{m}{a} = ma \left[\frac{2}{ar} - \frac{1}{mar^2} - \frac{1}{a^2} \right]. \quad \text{But } \frac{c^2}{ma} = 1 - e^2.$$

$$(104) \quad \therefore D_t r^2 = m \left[\frac{a^2 e^2 - (a-r)^2}{a r^2} \right].$$

If we let $k^2 = a(1 - e^2)$, and multiply (104) by $\frac{k^2}{e^2}$, we find

$$(105) \quad \frac{k^2 D_t r^2}{e^2} = 1 - \frac{(k^2 - r)^2}{e^2 r^2}.$$

By collecting equations (76) and (94) we have

$$(106) \quad c = x y' - y x',$$

$$\frac{m}{a} = \frac{2m}{r} - (x'^2 + y'^2) = \frac{2m}{r} - s'^2 = \frac{2m}{k^2} [1 + e \cos(\lambda + \pi)] - (x'^2 + y'^2),$$

in which $z' = 0$, in accordance with Art. 30; and $\lambda + \pi = \varphi$ are so chosen that π is the longitude of the perihelion. Since

$$\frac{k^2}{r} = 1 + e \cos(\lambda + \pi), \quad \cos(\lambda + \pi) = \frac{k^2 - r}{e r};$$

and hence

$$(107) \quad \sin(\lambda + \pi) = \sqrt{\left(1 - \frac{(k^2 - r)^2}{e^2 r^2}\right)} = \frac{k D_t r}{e},$$

by (105).

The manner in which we found equation (100) shows that $D_x A$, $D_y A$, $D_t A$, are partial differential coefficients. The first of equations (106) will then give

$$D_x c = -y, \quad D_y c = x,$$

and these values in (100), regarding $Q Z = 0$, give

$$(108) \quad D_t c = -Q X \cdot y + Q Y \cdot x.$$

The second of (106) gives $m D_x a = 2 a^2 x'$, $m D_y a = 2 a^2 y'$. These values in (100) give

$$(109) \quad D_t a = \frac{2 a^2}{m} [Q X \cdot D_t x + Q Y \cdot D_t y].$$

We also have, from (106),

$$D_x e = \frac{k^2 x'}{m \cos(\lambda + \pi)}, \quad D_y e = \frac{k^2 y'}{m \cos(\lambda + \pi)}.$$

These values, in (100), give

$$(110) \quad D_t e = \frac{k^2}{m \cos(\lambda + \pi)} [Q X \cdot D_t x + Q Y \cdot D_t y] = \frac{k^2 e r}{2 a^2 (k^2 - r)} \cdot D_t a.$$

For the variation of π , (106) gives

$$D_x \pi = -\frac{k^2 x'}{m e \sin(\lambda + \pi)} \quad D_y \pi = -\frac{k^2 y'}{m e \sin(\lambda + \pi)}.$$

These values, substituted in (100), and reducing by means of (107), we find

$$(111) \quad D_t \pi = -\frac{k}{m D_{t,r}} [Q X \cdot D_t x + Q Y \cdot D_t y] = -\frac{k}{2 a^2 D_{t,r}} \cdot D_t a.$$

From (59) we have

$$(112) \quad m T^2 = 4 \pi^2 a^3, \quad D_t T = \frac{6 \pi^2 a^2}{m T} \cdot D_t a.$$

The mean angular velocity, or *mean motion*, is

$$n = \frac{2 \pi}{T}, \quad n^2 = \frac{4 \pi^2}{T^2}, \quad n^2 a^3 = \frac{4 \pi^2 a^3}{T^2} = m,$$

by (59). Hence,

$$(113) \quad n^2 = m a^{-3}, \quad \text{and} \quad D_t n = -\frac{3 m}{2 n a^4} \cdot D_t a.$$

If we put M' for the mean longitude at a given epoch, and then $nt + M'$ will give the mean longitude for any time t , after; and, since π is the longitude of the perihelion, $nt + M' - \pi$ will give the mean anomaly at any time; and hence, by (78),

$$(114) \quad nt + M' - \pi = E - e \sin E.$$

If M' varies, it will be in consequence of the variation of π and e , since nt and E are variable quantities in elliptic motion. Hence,

$$D_t M' = D_t \pi - \sin E D_t e = D_t \pi - \sqrt{\left(1 - \left(\frac{a-r}{a}\right)^2\right)} D_t e.$$

By substituting for $D_t \pi$ and $D_t e$, their values as given by (110) and (111), we have

$$(115) \quad D_t M' = -\left[\frac{k}{2 a^2 D_{t,r}} + \frac{k^2 e r}{2 a^2 (k^2 - r)} \sqrt{\left(1 - \left(\frac{a-r}{a}\right)^2\right)}\right] D_t a.$$

32. To investigate the effects of the tangential force, we shall use the equation

$$(116) \quad \frac{m}{a} = \frac{2m}{k^2} [1 + e \cos(\lambda + \pi)] - S'^2.$$

For a , we have $D_s a = \frac{2a^2 s'}{m}$, and hence, by (103),

$$(117) \quad D_t a = \frac{2a^2}{m} \cdot Q S \cdot D_t s.$$

For e , we have $D_s e = \frac{k^2 s'}{m \cos(\lambda + \pi)} = \frac{k^2 e r s'}{m(k^2 - r)}$. Hence, by (103),

$$(118) \quad D_t e = \frac{k^2 e r}{m(k^2 - r)} \cdot Q S \cdot D_t s.$$

For π , we have $D_s \pi = -\frac{k^2 s'}{m e \sin(\lambda + \pi)} = -\frac{k^2 s'}{m D_t r}$. Hence,

$$(119) \quad D_t \pi = -\frac{k^2}{m D_t r} \cdot Q S \cdot D_t s.$$

$$(120) \quad D_t M' = -\frac{k^2}{m} \left[\frac{1}{D_t r} + \frac{e r}{k^2 - r} \sqrt{1 - \left(\frac{a-r}{a}\right)^2} \right] \cdot Q S \cdot D_t s.$$

The preceding equations expressing the differential coefficients of the elliptic elements with respect to the time, give the momentary variations of these elements, and without the aid of the integral calculus, they are capable of furnishing us accurate ideas respecting the action of the disturbing force.



NOTES AND QUERIES.

I. *Note on a Question in Interest.* — In the March number of the Monthly, Prof. HOYT cites the following from ROBINSON'S Algebra: — "A man borrowed \$ 60; what sum shall he pay daily to cancel the debt, principal and interest, in 60 days; interest at 10 per cent for

12 months of 30 days each?" The answers given in different editions are $\$1\frac{5}{800}$, and $\$1\frac{11}{200}$; and Prof. HOYT asks, "Is either of them *correct*, or is the true answer $\$1\frac{1}{200}$?"

Before answering this question, let us consider what rule or principle shall guide us in treating the commercial relations described in the problem. It is required to pay principal and interest. *What* interest? There are in common use two modes of adjusting the interest account, when an obligation is cancelled by partial payments. The one is described by the United States rule; the other by the Vermont Rule. According to the former, the interest to be cancelled is that which shall accrue on the unpaid parts of the principal during the intervals between the successive payments; according to the latter, the interest to be paid is the excess of the interest of the given sum for the whole time, above that accruing on the various payments from the dates at which they were severally made to the time of settlement. These two methods give different results. The problem under consideration involves precisely those commercial relations which require the application of one or the other of these rules. ROBINSON has adopted, in his solutions, neither of these rules, and both of his answers are erroneous. Prof. HOYT has based his solution on the Vermont rule, and has given the correct answer according to that rule. But it seems to me, that in all examples of this nature, where this condition is not defined in the problem, the United States rule should be applied. Not only has it the higher commercial authority, but the analysis by this principle is more pertinent and rational. The given sum is made to draw interest for the first interval. Then a payment is made, which cancels the interest due and a part of the principal. There is now left a balance of the principal on which interest will accrue for another interval; and so on.

To obtain a general solution, let p represent the principal, r the

interest of \$1 for one interval, y one of the equal payments, and n the number of payments. Then

$$(1) \quad y = \frac{p r (1+r)^n}{(1+r)^n - 1}, \text{ by the United States rule,}$$

$$(2) \quad y = \frac{2 p (1+r n)}{n(n-1)r + 2n} \text{ by the Vermont rule.}$$

The answer to this problem by equation (1) is \$ 1.0084984+.

It is but just to say that ROBINSON is not the only author in fault in respect to examples of this kind. In DAVIES's University Arithmetic we find the following:—

“A farmer bought a piece of land for \$ 1,500, and agreed to pay principal and interest in five equal annual instalments; how much was the annual payment?”

A reference to the Key shows that 7 per cent interest is understood. The answer given is \$ 360.56. Now this answer cannot be verified by any recognized rule for partial payments. The annual payment required to cancel an obligation of \$ 1,500 in five years is \$ 366.08, interest reckoned by the United States rule, and \$ 342.10, interest reckoned by the Vermont rule.—Prof. J. C. PORTER, Liberal Institute, Clinton, New York.

II. *Note on the General Theory of Equations.*—In developing the general theory of equations, it is usual to multiply together the transposed equations formed by x with each of its assigned values, and to treat x in the product, though of different values, as one quantity. The student is often perplexed by this process. Why set down as x^2 the product of x having one value, by x having another value? Why unite the several terms containing the first power of x in the product, when in fact x has different values in those terms? And why will each one of the assigned values of x

verify the resulting equation by being substituted throughout, when the terms which are written as powers of x are the products of factors originally different, or the sums of such products?

In order to explain this, let us at first keep all the monomial factors distinct. Multiplying together $x + a = 0$, $x' + b = 0$, $x'' + c = 0$, &c., we obtain the successive equations,

$$\begin{aligned}(x + a)(x' + b) &= 0. \\ (x + a)(x' + b)(x'' + c) &= 0. \\ (x + a)(x' + b)(x'' + c)(x''' + d) &= 0, \text{ \&c.}\end{aligned}$$

Now it is evident that whatever be the number of binomial factors used as above, we may substitute x for x' , x'' , &c., or any one of these for all the others, without destroying the equation; for in each case one of the factors, and, consequently, the whole product, will still remain equal to zero.

For an example, take the second of the above equations. Expanding this, we obtain an equation having eight terms. But since $x + a = 0$, we have $(x + a)(x + b)(x + c) = 0$; and since $x' + b = 0$, $(x' + a)(x' + b)(x' + c) = 0$; and since $x'' + c = 0$, $(x'' + a)(x'' + b)(x'' + c) = 0$. Expanding these and uniting like terms, we get from one equation three numerically distinct equations, of four terms each.

$$\begin{aligned}x^3 + Ax^2 + Bx + C &= 0. \\ x'^3 + Ax'^2 + Bx' + C &= 0. \\ x''^3 + Ax''^2 + Bx'' + C &= 0.\end{aligned}$$

All these are obviously contained in the following general formula, where X is to be understood as having three values:—

$$X^3 + AX^2 + BX + C = 0.$$

Thus it appears that the equation obtained by blending x , x' , x'' ,

&c. in the product, though different from that which we should obtain by keeping them distinct, is nevertheless a true equation, and one which each of the originally assigned values of x will verify.
— Prof E. W. EVANS, Marietta College, Ohio.

III. *Trigonometrical Formulas.* — The values of $\sin (A \pm B)$, $\cos (A \pm B)$ in terms of the sines and cosines of A and B may be deduced as follows: Take two triangles whose sides are a, b, c , and opposite angles A, B, C , and suppose the angle A in one of them to be obtuse. From the vertical angle C , drop a perpendicular, d , on the base in one case, and on the base produced in the other. The triangles stand on the base AB , and D is the foot of the perpendicular. If $A = CAD$ and $B = CBD$ in both cases, then $\sin C = \sin (A + B)$ in the first, and $\sin C = \sin (A - B)$ in the second, and

$$\begin{aligned}\sin C = \sin (A \pm B) &= \frac{dc}{ab} = \frac{d(DB \pm AD)}{ab} = \frac{d}{b} \cdot \frac{DB}{a} \pm \frac{d}{a} \cdot \frac{AD}{b} \\ &= \sin A \cos B \pm \sin B \cos A,\end{aligned}$$

since, in both cases,

$$\sin A = \frac{d}{b}, \quad \sin B = \frac{d}{a}, \quad \cos A = \frac{AD}{b}, \quad \cos B = \frac{DB}{a}.$$

In the first case, $\cos C = -\cos (A + B)$, and, in the second, $\cos C = \cos (A - B)$. But $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$, and therefore

$$\begin{aligned}\cos C = \cos (A \pm B) &= \mp \frac{a^2 + b^2 - c^2}{2ab} = \frac{AD}{b} \cdot \frac{BD}{a} \mp \frac{d}{b} \cdot \frac{d}{a} \\ &= \cos A \cos B \mp \sin A \sin B,\end{aligned}$$

$$a^2 + AD^2, \quad a^2 = d^2 + BD^2, \quad c^2 = (BD \pm AD)^2.$$

MOORE, McKendree College, Lebanon, Illinois.

Mathematical Monthly Notices.

The Claim of LEIBNITZ to the Invention of the Differential Calculus, by DR. H. SLOMAN. Translated from the German with considerable Alterations and new Addenda by the Author. Cambridge: MACMILLAN & Co., and 23 Henrietta Street, Covent Garden, London. 1860. 4to. pp. 157.

A few years ago PROFESSOR DE MORGAN remarked, that "all the notions as to what is fair and what is unfair in controversy have undergone much change since the commencement of the last century." This remark may be true as to the world in general, but it is evidently false as to DR. SLOMAN. Into the great controversy concerning the invention of the Differential Calculus, which of late years has had so much new interest thrown around it, our author enters with a bitterness that KEILL and FATIO DE DUILLIER might have envied. He espouses the Newtonian side of the contest, and endeavors to turn back the tide which has of late been so steadily setting towards a complete vindication of the fame of LEIBNITZ. The researches of GERHARDT, DE MORGAN, and others, and the new edition of the *Commercium Epistolicum* by BIOT and LEFORT, were fast establishing the claim of LEIBNITZ to an independent invention of the Differential Calculus, as well as to the sole invention of the notation that gave to it its vast power. But while every true man was rejoicing to see this vindication of a great intellect, our author enters the field with very different feelings. With very little in the way of new facts, and that little unimportant, his book is filled with misrepresentations, garbled extracts, mean insinuations, and base perversions. The only thing it exhibits fairly, it exhibits undesignedly, namely, the moral character of its author, showing it to be of that low type that labors to put the worst construction on every word and act of an opponent.

We do not purpose to enter into an examination of this book in detail. To do so would involve a review of the whole controversy, and would, moreover, ascribe to the work an importance which it does not deserve. We shall, however, examine a preliminary page, which purports to give extracts from GERHARDT, EDLESTON, and LEIBNITZ, and which stands thus:—

GERHARDT: "So it is Leibnitz — — to whom we owe this. The Chapter in which the question respecting the first discoverer of the higher Analysis has till now been agitated, now disappears from the history of the mathematical sciences. The battle of more than one hundred years, about the first discoverer of the Differential Calculus, is now at its end." <i>Gerhardt</i>, II. page 62.	GERHARDT: "So ist es Leibnitz — — dem wir dies verdanken. Das Capitel, in welchem die Frage über den ersten Entdecker der höheren Analysis bisher erörtert wurde, verschwindet fortan aus der Geschichte der mathematischen Wissenschaften. Der mehr als hundert jährige Streit über den ersten Entdecker der Differenzialrechnung ist zu Ende." <i>Gerh. Abh.</i> II. S. 62.
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EDLESTON: Synoptical view of Newton's life:

1666 Octob. Small tract on fluxions and fluents with their applications to a variety of problems on tangents, curvature, areas, lengths, and centres of gravity of curves.

1666 November. Small tract similar to the preceding but apparently more comprehensive. (Notation by points in first and second fluxions, Basis of his larger tract of 1671.) *Edleston* in his Correspondence of Sir I. Newton, 1850, page 21.

LEIBNITZ: Cum Parisios apulisset anno Christi 1672 eram ergo — — in pene dixerim superba Matheseos ignorantia. *Leibnitz* in *Gerhardt's Pamphlet*, I. page 29 et 30, line 2.

The reader would suppose from this, that GERHARDT here claims that LEIBNITZ was "the first discoverer of the Differential Calculus," which claim is disposed of by showing, in the succeeding extracts, that NEWTON had his method of Fluxions in 1666, six years before the time

when LEIBNITZ declared his ignorance of Mathematics. We pass over the mistakes in the last extract, of *apulissem* for *appulissem*, and *ergo* for *ego*, and also the bringing together into one sentence of the parts of widely separated sentences, because it happens that here the sense is not essentially changed. But it is otherwise with the first extract. The first words, "So ist es LEIBNITZ," are not in GERHARDT; the next four, "dem wir dies verdanken," are altered from a sentence on the preceding page, and the remainder of the extract is part of a sentence separated from the rest by more than half a page. The meaning of GERHARDT is, by this ingenious process, entirely perverted. The passage occurs in a chapter headed, "*The Discovery of the Algorithm of the Higher Analysis by LEIBNITZ*," and has reference to the system of notation invented by him. No one could mistake this; for GERHARDT's words in the first sentence quoted from are, "the notation which we owe to Leibnitz" (*die Bezeichnungsweise die wir Leibnitz verdanken*). GERHARDT does not deny, and no one, we presume, denies, that NEWTON had made the discovery of what was afterwards called his Method of Fluxions (but without any general notation) several years before LEIBNITZ began the study of the higher mathematics; and his object in this place is simply to claim for LEIBNITZ the invention of his system of notation. After showing from his manuscripts the successive steps in the development of the subject, as they presented themselves to the mind of LEIBNITZ, GERHARDT stops a moment to consider the importance of the invention, and closes with the opinion, that every suspicion of plagiarism on the part of LEIBNITZ must now vanish, and, therefore, that the long controversy may now be considered at an end. We give below a translation of the whole passage, — interesting in itself, — that our readers may judge for themselves how fairly the extracts of Dr. SLOMAN are made.

"This is the origin of that notation to which the Higher Analysis owes its growth and wonderful perfection, — and the twenty-ninth of October, 1675, is the memorable day that saw its birth. The importance of the subject demands that we should pause an instant, in order to rightly estimate this significant moment. The bold speculations of KEPLER and the *Method of Indivisibles* of CAVALIERI had furnished the means of treating more easily than before the problems of higher geometry. In addition, WALLIS, in his *Arithmetica Infinitorum*, had shown how the general methods of representing magnitudes, introduced by VIETA and DESCARTES, could be used in quadratures and cubatures. By these means a great many particular results were reached that are now obtained by the Higher Analysis; but there was wanting a common bond to free them from their isolation, and unite them under a single point of view. This the natural and most happy notation which we owe to LEIBNITZ effected at a single stroke. It has been universally acknowledged by the great mathematicians of succeeding times, that the Algorithm of LEIBNITZ called forth the surprisingly rapid progress that was made in enlarging and applying the Higher Analysis, and brought about the completion of that truly noble edifice, to survey which and to explore whose various parts are given only to him in whom eminent ability is found united with the highest power of attention and unwearied perseverance. It is not to be overlooked, that LEIBNITZ, led by his lucky genius, came almost accidentally upon the introduction of the algorithm of the Higher Analysis; nor, on the other hand, can it be denied, that being prepared for it by his earlier studies, and surveying the field of science, he recognized at once, though not in its full extent, the importance of the happy discovery, and knew how to employ it in the most skilful manner. This, however, every one must openly concede, — that LEIBNITZ, in introducing his new algorithm, proceeded independently throughout, without even the slightest external influence; an unprejudiced examination of his manuscripts forces us to this conclusion. All doubt, every suspicion of plagiarism, is removed by the manuscripts which are here for the first time connectedly published, and the chapter in which the question respecting the first discoverer of the Higher Analysis has been till now agitated, henceforth disappears from the history of the mathematical sciences. The battle of more than one hundred years about the first discoverer of the Differential Calculus is now at an end."

Let us hope that the spirit which animates the work of DR. SLOMAN will have few imitators. Let us rather cherish and defend the fame of both NEWTON and LEIBNITZ, recollecting that the honor to our race is doubled, if two, instead of one, are shown to be capable of a great invention. Let us be ready to rejoice whenever the impartial hand of later research reveals a new brightness in those great names which the envy and malice of contemporaries sometimes obscure.

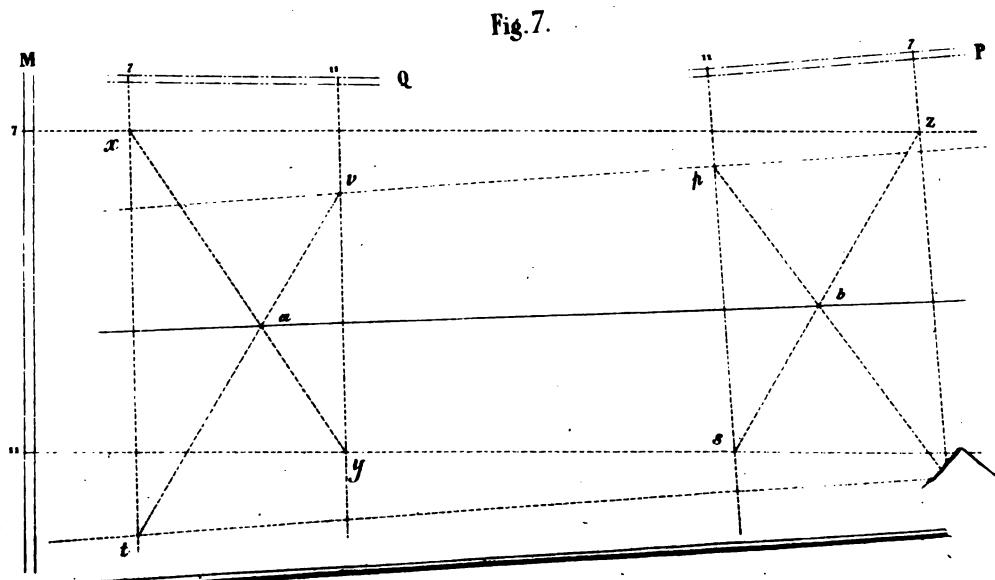
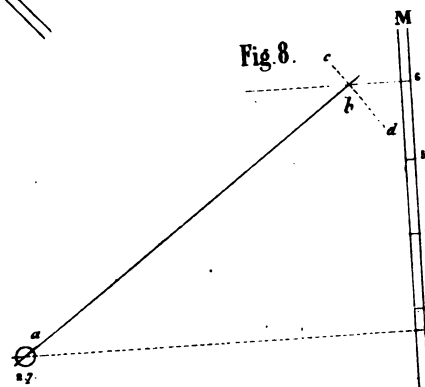
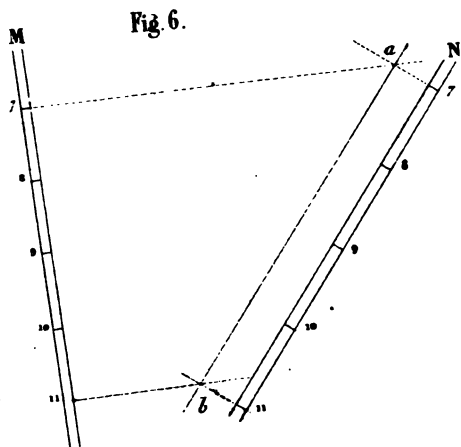
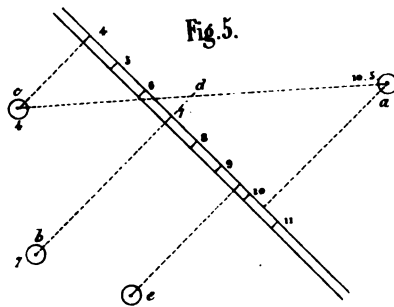
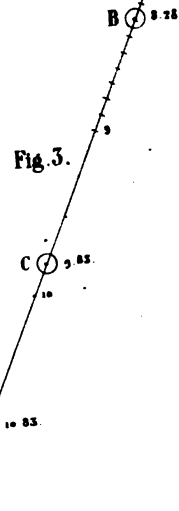
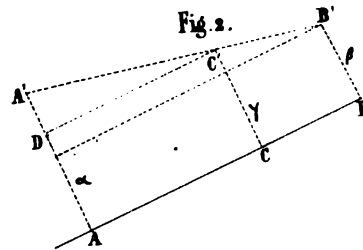
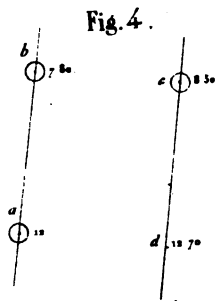
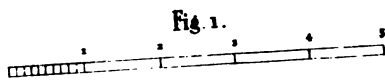


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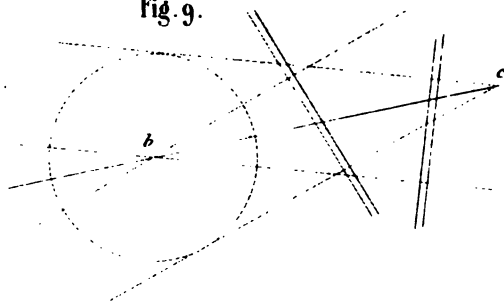


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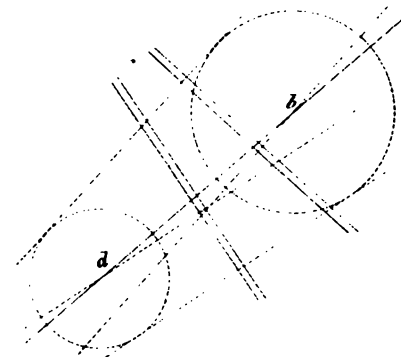


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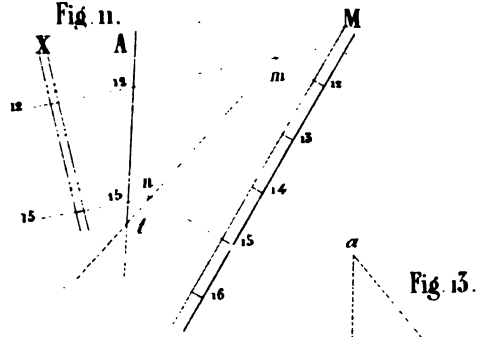


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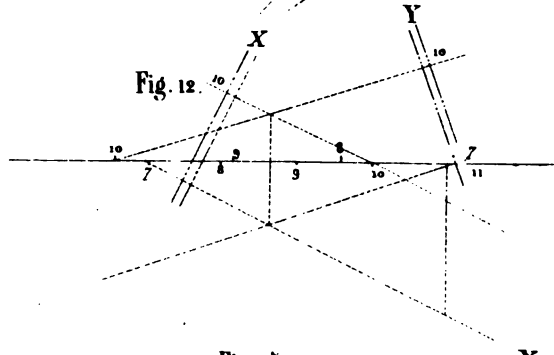


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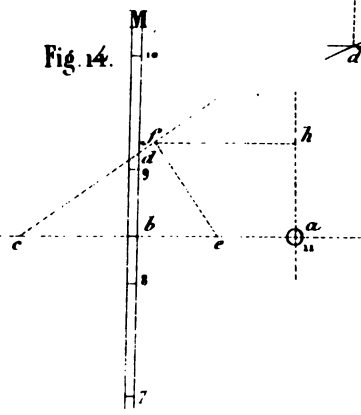


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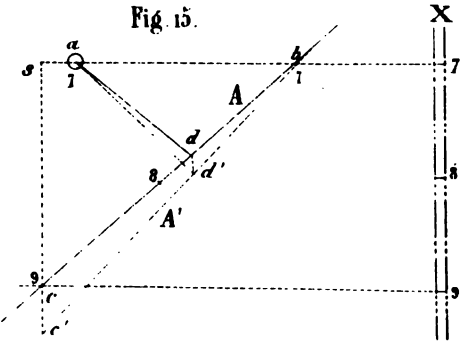


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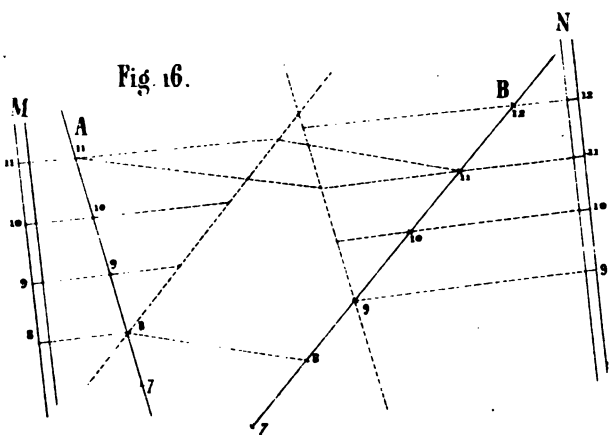


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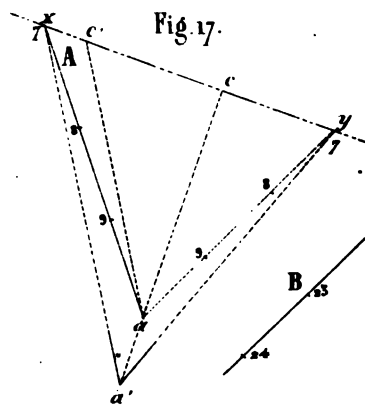


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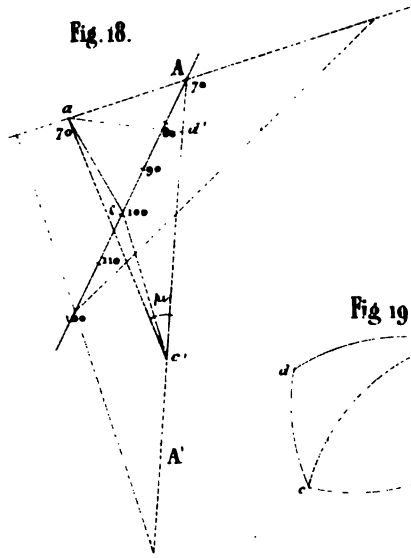


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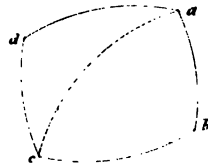


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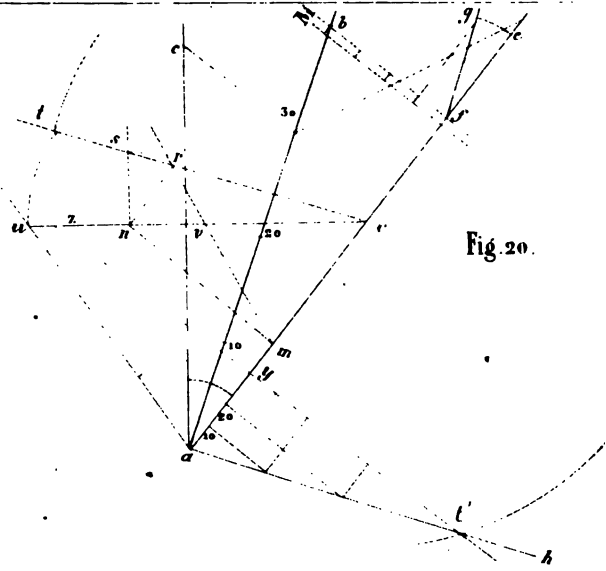


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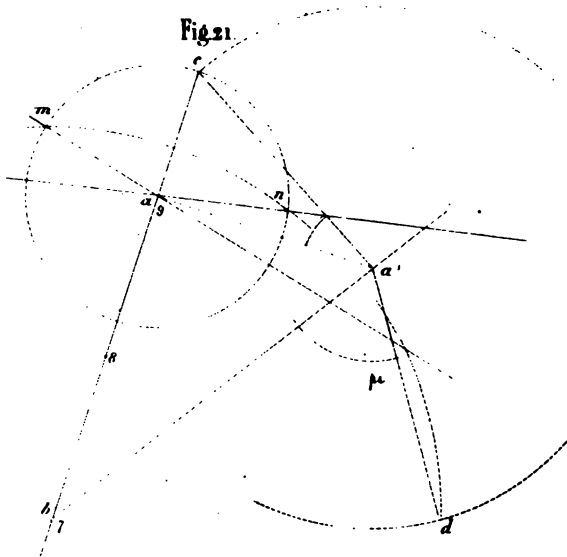


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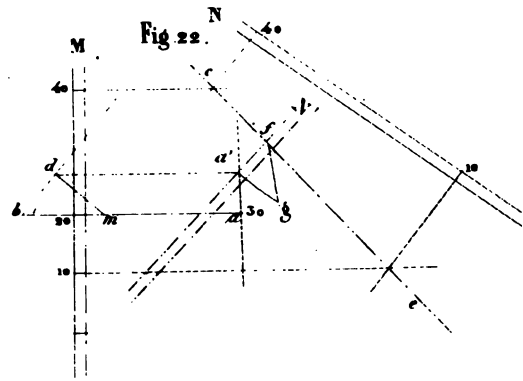


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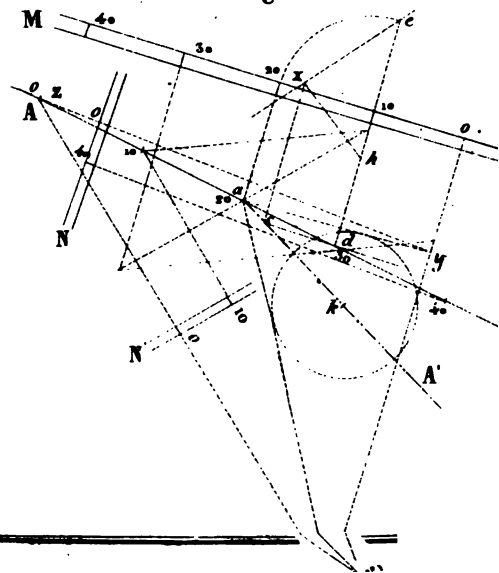


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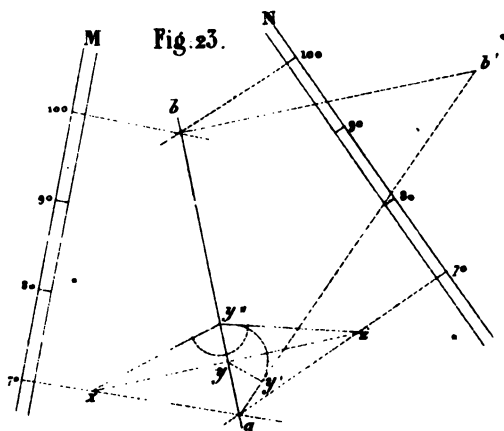


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